Bounds for Real Roots and Applications to Orthogonal Polynomials

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Introduction

- We obtain new inequalities on the real roots of a univariate polynomial with real coefficients.
- We derive estimates for the largest positive root, a key step for real root isolation. We discuss the case of classic orthogonal polynomials.
- We compute upper bounds for the roots of orthogonal polynomials using new inequalities derived from the differential equations satisfied by these polynomials.
- We compare our results with those obtained by other methods.

Bounds for Real Polynomial Roots

The computation of the real roots of univariate polynomials with real coefficients is based on their isolation. To isolate the real positive roots, it is sufficient to estimate the smallest positive root (cf. [2] and [21]). This can be achieved if we are able to compute accurate estimates for the largest positive root.

Computation of the Largest Positive Root

Several bounds exist for the absolute values of the roots of a univariate polynomial with complex coefficients (see, for example, [15]). These bounds are expressed as functions of the degree and of the coefficients, and naturally they can be used also for the roots (real or complex) of polynomials with real coefficients. However, for the real roots of polynomials with real coefficients there also exist some specific bounds. In particular, some bounds for the positive roots are known, the first of which were obtained by Lagrange [11] and Cauchy [5]. We briefly survey here the most often used bounds for positive roots and discuss their efficiency in particular cases, emphasizing the classes of orthogonal polynomials. We then obtain extensions of a bound of Lagrange, and derive a result also valid for positive roots smaller than 1.

A bound of Lagrange

Theorem 1 (Lagrange) *Let*
$P(X) = a_0 X^d + \cdots + a_m X^{d-m} - a_{m+1} X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]$,
with all $a_i \geq 0$*,* $a_0, a_{m+1} > 0$ *. Let*

$$A \;=\; \max\left\{a_i \,;\, \text{coeff}\,(X^{d-i}) \;<\; 0\right\} \,.$$

The number

$$1 \,+\, \left(\frac{A}{a_0}\right)^{1/(m+1)}$$

is an upper bound for the positive roots of P *.*

The bound from Theorem 1 is one of the most popular (cf. H. Hong [8]), however it gives only bounds larger than one. For polynomials with subunitary real roots, it is recommended to use the bounds of Kioustelidis [9] or Ştefănescu [18]. A discussion on the efficiency of these results can be found in Akritas–Strzeboñski–Vigklas [2] and Akritas–Vigklas [3].

Extensions of the bound of Lagrange

We give a result that extends the bound $L_1(P)$ of Lagrange.

Theorem 2 *Let*
$P(X) = a_0X^d + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]\,,$
with all $a_i \geq 0$*,* $a_0, a_{m+1} > 0$ *. Let*

$$A \;=\; \max\left\{a_i\,;\,\text{coeff}\,(X^{d-i}) \;<\; 0\right\}\;.$$

The number

$$\left\{\begin{array}{l} 1 \quad + \quad \max\left\{\left(\dfrac{pA}{a_0+\cdots+a_s}\right)^{1/(m-s+1)}\right., \\[2ex] \qquad\qquad \left(\dfrac{qA}{sa_0+\cdots+2a_{s-2}+a_{s-1}}\right)^{1/(m-s+2)}, \\[2ex] \qquad\qquad \left.\left(\dfrac{2rA}{s(s-1)a_0+(s-1)(s-2)a_1+\cdots+2a_{s-2}}\right)^{1/(m-s+3)}\right\} \end{array}\right. \tag{1}$$

is an upper bound for the positive roots of P *for any* $s \in \{2,3,\ldots,m\}$ *and* $p \geq 0\,, q \geq 0\,, r \geq 0$ *such that* $p+q+r \,=\, 1\,.$

The proof of Theorem 2 is similar to that of our Theorem 1 in [19].

Particular cases of Theorem 2

1. For $p = 1$, $q = r = 0$, we obtain the bound

$$1 + \left(\frac{A}{a_0 + \cdots + a_s} \right)^{1/(m-s+1)} .$$

This bound is also valid for $s = 0$ and $s = 1$. For $s = 0$, it reduces to the bound $L_1(P)$ of Lagrange.

2. For $p = q = r = 1/3$, we obtain Theorem 1 from [19].

Particular cases of Theorem 2 (contd.)

3. For $p = q = 1/4$, $r = 1/2$, we obtain

$$1 \quad + \quad \max\left\{ \left(\frac{A}{4(a_0 + \cdots + a_s)} \right)^{1/(m-s+1)} , \right.$$

$$\left(\frac{A}{4(sa_0 + \cdots + 2a_{s-2} + a_{s-1})} \right)^{1/(m-s+2)} ,$$

$$\left(\frac{A}{s(s-1)a_0 + (s-1)(s-2)a_1 + \cdots + 2a_{s-2}} \right)^{1/(m-s+3)} .$$

Particular cases of Theorem 2 (contd.)

4. For $p = q = \dfrac{1}{2}$, $r = 0$, we obtain

$$1 \quad + \quad \max\left\{ \left(\frac{A}{2(a_0 + \cdots + a_s)} \right)^{1/(m-s+1)} \right. ,$$

$$\left. \left(\frac{A}{2(sa_0 + \cdots + 2a_{s-2} + a_{s-1})} \right)^{1/(m-s+2)} \right\} ,$$

which is Theorem 3 from [18]. This bound is also valid for $s = 0$.

Example

Let

$$\begin{aligned} P_1(X) &= X^{17} + X^{13} + X^{12} + X^9 + 3X^8 + 2X^7 + X^6 - 5X^4 \\ &\quad +X^3 - 4X^2 - 6, \\ P_2(X) &= X^{13} + X^{12} + X^9 + 3X^8 + 2X^7 + X^6 - 6X^4 + X^3 \\ &\quad -4X^2 - 7\,. \end{aligned}$$

We denote:

$B(P) = B(m, s, p, q, r)$, the bound given by Theorem 1

$L_1(P)$ = the bound of Lagrange (Theorem 1)

LPR = the largest positive root

For P_1 we have $A = 6$ and $m = 11$, and for P_2 we have $A = 7$ and $m = 6$.

Example (contd.)

P	s	p	q	r	$B(P)$	$L_1(P)$	LPR
P_1	8	0.5	0.5	0	13.89	2.161	1.53
P_1	2	0.5	0.5	0	3.15	2.161	1.53
P_1	1	0.5	0.5	0	2.00	2.161	1.53
P_1	8	0.4	0.3	0.3	64.78	2.161	1.53
P_1	2	0.2	0.6	0.2	3.25	2.161	1.53
P_2	7	0.5	0.5	0	8.25	2.232	1.075
P_2	3	0.4	0.6	0	7.18	2.232	1.075
P_2	3	0.5	0.5	0	6.85	2.232	1.075
P_2	1	0.4	0.6	0	3.07	2.232	1.075
P_2	5	0.4	0.3	0.3	26.2	2.232	1.075
P_2	2	0.4	0.3	0.3	4.02	2.232	1.075
P_2	2	0.6	0.2	0.2	3.84	2.232	1.075

Comparison with the bound of Lagrange

We compare the bound given by Theorem 2 with that of Lagrange

$$L_1(P) = 1 + \left(\frac{A}{a_0}\right)^{1/(m+1)} .$$

We consider $p = q = 0.25$, $r = 0.5$ and $s = 2$ in Theorem 2. Then

$$\begin{aligned} B(P) \; = \; 1 \quad + \quad \max\Bigg\{ & \left(\tfrac{A}{4(a_0+a_1+a_2)}\right)^{1/(m-1)}, \\ & \left(\tfrac{A}{4(2a_0+a_1)}\right)^{1/m}, \left(\tfrac{A}{2a_0}\right)^{1/(m+1)} \Bigg\} . \end{aligned}$$

We can see which of the bounds $B(P)$ and $L_1(P)$ is better by looking to the size of A with respect to a_0, a_1, a_2 and m.

Comparison with the bound of Lagrange (contd.)

We obtain:

- $B(P) < L_1(P)$ if

$$A < \min\left\{\frac{\Big(4(a_0+a_1+a_2)\Big)^{(m+1)/2}}{a_0^{(m-1)/2}}, \frac{4^{m+1}(2a_0+a_1)^{m+1}}{a_0^m}\right\}$$

- $B(P) > L_1(P)$ if

$$A > \max\left\{\frac{\Big(4(a_0+a_1+a_2)\Big)^{(m+1)/2}}{a_0^{(m-1)/2}}, \frac{4^{m+1}(2a_0+a_1)^{m+1}}{a_0^m}\right\}.$$

Example

Let

$$P(X) = X^d + 3X^{d-1} + X^{d-2} + 0.001\,X^{d-3} + 0.0003\,X^{d-4}$$
$$-AX^4 - AX^3 - AX - A + 1,$$

with $A > 0$. Then we have:

d	A	$L_1(P)$	$B(P)$	LPR
10	3	2.201	2.069	1.146
11	3	2.201	2.069	1.126
8	4	2.256	2.122	1.287
9	4	2.256	2.122	1.230
10	4	2.256	2.122	1.193
10	20^6	20.999	43.294	19.687

Other Bounds for Positive Roots

Note that the bound $L_1(P)$ of Lagrange and its extensions give only numbers greater than one, so they cannot be used for some classes of polynomials. For example, the roots of Legendre orthogonal polynomials are subunitary.

Kioustelidis, 1986

J. B. Kioustelidis [9] gives the following upper bound for the positive real roots:

Theorem 3 (Kioustelidis) *Let* $P(X) = X^d - b_1 X^{d-m_1} - \cdots - b_k X^{d-m_k} + g(X)$*, with* $g(X)$ *having positive coefficients and* $b_1 > 0, \ldots, b_k > 0$*. The number*

$$K(P) \;=\; 2 \cdot \max\{b_1^{1/m_1}, \ldots, b_k^{1/m_k}\}$$

is an upper bound for the positive roots of P.

Ştefănescu, 2005

For polynomials with an even number of variations of sign, we proposed in [18] another bound. Our method can be applied also to polynomials having at least one sign variation.

Theorem 4 *Let $P(X) \in \mathbb{R}[X]$ and suppose that P has at least one sign variation. If*

$$P(X) \;=\; c_1X^{d_1} - b_1X^{m_1} + c_2X^{d_2} - b_2X^{m_2} + \cdots + c_kX^{d_k} - b_kX^{m_k} + g(X)\,,$$

with $g(X) \in \mathbb{R}_+[X]$, $c_i > 0$, $b_i > 0$, $d_i > m_i$ for all i, the number

$$S(P) = \max\left\{\left(\frac{b_1}{c_1}\right)^{1/(d_1-m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k-m_k)}\right\}$$

is an upper bound for the positive roots of P.

Remarks

We obtained in [18], Theorem 2, another version of Theorem 4, under the additional assumption that the polynomial has an even number of sign variations and that $d_i > m_i > d_{i+1}$ for all i. But any polynomial having at least one sign variation can be represented (not uniquely!) as

$$P(X) = c_1X^{d_1} - b_1X^{m_1} + c_2X^{d_2} - b_2X^{m_2} + \cdots + c_kX^{d_k} - b_kX^{m_k} + g(X)\,,$$

with $g(X) \in \mathbb{R}_+[X]$, $c_i > 0$, $b_i > 0$, $d_i > m_i$ for all i.

In 2006, Akritas *et al.* presented in [2] a result based on Theorem 2 from [18]. Their approach to adapt our theorem to any polynomial with sign variations uses a representation (also not unique!)

$$P(X) = \sum_{i=1}^{m} (q_{2i-1}(X) - q_{2i}(X)) + g(X),$$

where all q_j and g have positive coefficients, and some inequalities among the degrees of the monomials of q_{2i-1} and q_{2i} are satisfied.

Remarks (contd.)

Our Theorem 2 from [18] and the extensions of Akritas *et al.* were implemented in [2] and [3].

If a polynomial $P \in \mathbb{R}[X]$ has all real positive roots in the interval $(0, 1)$, using the transformation $x \to 1/x$ we obtain a polynomial — called the reciprocal polynomial — with positive roots greater than one. If we compute a bound ub for the positive roots of the reciprocal polynomial, the number $lb = 1/ub$ will be a lower bound for the positive roots of the initial polynomial P. This process can be applied to any real polynomial with positive roots, and is a key step in the Continued Fraction real root isolation algorithm (see [2] and [21]).

Lagrange, 1769

In some special cases the following other bound of Lagrange is useful:

Theorem 5 *Let F be a nonconstant monic polynomial of degree n over $\mathbb{R}$ and let $\{a_j\,;\, j \in J\}$ be the set of its negative coefficients. Then an upper bound for the positive real roots of F is given by the sum of the largest and the second largest numbers in the set*

$$\left\{ \sqrt[j]{|a_j|}\,;\, j \in J \right\} .$$

Theorem 5 can be extended to absolute values of polynomials with complex coefficients (see M. Mignotte–D. Ştefănescu [14]).

Notation

- The bounds of Lagrange from Theorems 1 and 5 will be denoted by $L_1(P)$, respectively $L_2(P)$.
- The bound of Kioustelidis from Theorem 3 is denoted by $K(P)$.

Example

Let $P(X) = 2X^7 - 3X^4 - X^3 - 2X + 1 \in \mathbb{R}[X]$. The polynomial P does not fulfill the assumption $d_i > m_i > d_{i+1}$ for all i from Theorem 2 in [18]. However, after the decomposition of the leading coefficient in a sum of positive numbers, Theorem 4 can be applied.

We use the following two representations:

$$\begin{aligned} P(X) &= P_1(X) \\ &= (X^7 - 3X^4) + (0.5\,X^7 - X^3) + (0.5\,X^7 - 2X) + 1, \end{aligned}$$

$$\begin{aligned} P(X) &= P_2(X) \\ &= (1.1\,X^7 - 3X^4) + (0.4\,X^7 - X^3) + (0.5\,X^7 - 2X) + 1. \end{aligned}$$

Example (contd.)

We denote $S_j(P) = S(P_j)$ for $j = 1, 2$, and obtain the bounds

$$S_1(P) = 1.442\,, \quad S_2(P) = 1.397\,.$$

The largest positive root of P is 1.295.

Other bounds give

$$K(P) = 2.289\,, \quad L_1(P) = 2.404\,, \quad L_2(P) = 2.214\,.$$

Both $S_1(P)$ and $S_2(P)$ are smaller than $L_1(P)$, $L_2(P)$ and $K(P)$.

Bounds for Roots of Orthogonal Polynomials

Classical orthogonal polynomials have real coefficients and all their zeros are real, distinct, simple and located in the interval of orthogonality.

We first evaluate the largest positive roots of classical orthogonal polynomials using our previous results and a bound considered by van der Sluis in [17]. We also obtain new bounds using properties of of the differential equations which they satisfy. These new bounds will be compared with known bounds.

The polynomials P_n, L_n, T_n and U_n

The orthogonal polynomials of Legendre, Laguerre and Chebyshev of first and second kind:

$$
\begin{aligned}
P_n(X) &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} X^{n-2k}, \\
L_n(X) &= \sum_{k=0}^{n} \binom{n}{n-k} \frac{(-1)^k}{k!} X^k, \\
T_n(X) &= \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k \, 2^{n-2k}}{n-k} \binom{n-k}{k} X^{n-2k}, \\
U_n(X) &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \, 2^{n-2k} \binom{n-k}{k} X^{n-2k}.
\end{aligned}
$$

Bounds for P_n, L_n, T_n and U_n

Proposition 6 *Let P_n, L_n, T_n and U_n be the orthogonal polynomials of degree n of Legendre, respectively Laguerre and Chebyshev of first and second kind. We have*

i. The number $S(P_n) = \sqrt{\dfrac{n(n-1)}{2(2n-1)}}$ is an upper bound for the roots of P_n.

ii. The number $S(L_n) = n^2$ is an upper bound for the roots of L_n.

iii. The number $S(T_n) = \dfrac{\sqrt{n}}{2}$ is an upper bound for the roots of T_n.

iv. The number $S(U_n) = \dfrac{\sqrt{n-1}}{2}$ is an upper bound for the roots of U_n.

The Bound of Newton

Since orthogonal polynomials are hyperbolic polynomials (i.e., all their roots are real numbers), for the estimation of their largest positive root we can also use the bounds given by van der Sluis [17]. He considers monic univariate polynomials

$$P(X) \,=\, X^n + a_1 X^{n-1} + a_2 X^{n-2} + \cdots + a_n \in \mathbb{R}[X]$$

and mentions the following upper bound for the roots in the hyperbolic case:

$$Nw(P) \,=\, \sqrt{a_1^2 - 2a_2}\,.$$

For orthogonal polynomials Newton's bound gives

The Bound of Newton (contd.)

Proposition 7 *Let P_n, L_n, T_n and U_n be the orthogonal polynomials of degree n of Legendre, respectively Laguerre and Chebyshev of first and second kind. We have*

i. The number $Nw(P_n) = \sqrt{\dfrac{2(2n-2)!}{(n-1)!(n-2)!}}$ is an upper bound for the roots of P_n .

ii. The number $Nw(L_n) = \sqrt{n^4 - n^2(n-1)^2}$ is an upper bound for the roots of L_n .

iii. The number $Nw(T_n) = 2^{(n-1)/2}$ is an upper bound for the roots of T_n .

iv. The number $Nw(U_n) = \sqrt{(n-1)2^{n-1}}$ is an upper bound for the roots of U_n .

Comparisons on Orthogonal Polynomials

In the following tables we denote by L_1 the bound of Lagrange from Theorem 1, by K the bound of Kioustelidis, by S our bound from [18], by Nw the bound of Newton, and by LPR the largest positive root of the polynomial P.

We used the `gp-pari` package for computing the entries in the tables.

I. Bounds for Zeros of Legendre Polynomials

n	$L_1(P)$	K(P)	S(P)	Nw	LPR
5	2.05	2.10	1.054	141.98	0.901
8	2.367	2.73	1.366	157822.9	0.960
15	2.95	3.80	1.902	2.08×10^{14}	0.987
50	47.043	7.035	3.517	1.96×10^{76}	0.9988
120	26868.98	10.931.97	5.465	1.091×10^{231}	0.9998

II. Bounds for Zeros of Laguerre Polynomials

n	$L_1(P)$	K(P)	S(P)	Nw(P)	LPR
5	600	25	25	15.0	12.61
8	376321.0	64	25	30.983	22.86
15	7.44×10^{13}	225	225	80.777	48.026
50	6.027×10^{68}	2500	2500	497.49	180.698
120	1.94×10^{206}	14400	14400	1855.15	487.696

II. Bounds for Zeros of Chebyshev Polynomials of First Kind

n	$L_1(P)$	K(P)	S(P)	Nw	LPR
5	2.118	2.236	1.118	4.0	0.951
8	2.41	2.83	1.41	11.313	0.994
15	3.072	3.872	1.936	128.0	0.994
50	48.822	7.416	3.708	2.37×10^7	0.9995
120	27917.33	10.00	5.00	8.15^{17}	0.99991

IV. Bounds for Zeros of Chebyshev Polynomials of Second Kind

n	$L_1(P)$	K(P)	S(P)	Nw(P)	LPR
5	2.00	2.00	1.00	8.0	0.87
8	2.322	2.83	1.41	29.933	0.994
15	2.87	3.74	1.87	478.932	0.98
50	45.348	9.96	4.98	1.66×10^8	0.9981
120	25864.44	9.96	4.98	8.89×10^{18}	0.9996

Note that for Legendre and Chebyshev polynomials we have $K(P) = 2\,S(P)$.

Remarks

Other comparisons on roots of orthogonal polynomials were obtained by Akritas *et al.* in [3]. They consider the bounds of Cauchy and Lagrange, and also cite their result derived from our result in [18]. Obviously, in the case of classical orthogonal polynomials there exist an even number of sign variations, and thus Akritas *et al.* apply, in fact, our theorem.

We note that Newton bound gives the best results for Laguerre polynomials. Better estimates can be derived using the Hessian of Laguerre.

Bounds derived through the Hessian of Laguerre

Another approach for estimating the largest positive root of an orthogonal polynomial is the study of inequalities derived from the positivity of the Hessian associated to an orthogonal polynomial. They will allow us to obtain better bounds than known estimations.

If we consider

$$f(X) \;=\; \sum_{j=1}^{n} a_j\, X^j \,,$$

a univariate polynomial with real coefficients, its *Hessian* is

$$\mathrm{H}\,(f) \;=\; (n-1)^2\, f'^2 - n(n-1)\, f f' \;\geq\; 0\,.$$

The Hessian was introduced by Laguerre [12], who proved that $\mathrm{H}\,(f) \geq 0$.

Laguerre's Inequality

Let now $f \in \mathbb{R}[X]$ be a polynomial of degree $n \geq 2$ that satisfies the second–order differential equation

$$p(x)\, y'' + q(x)\, y' + r(x)\, y \;=\; 0\,, \tag{2}$$

with p, q and r univariate polynomials with real coefficients, $p(x) \neq 0$. We recall the following

Theorem 8 (Laguerre) *If all the roots of f are simple and real, we have*

$$4(n-1)\Big(p(\alpha)r(\alpha)+p(\alpha)q'(\alpha)-p'(\alpha)q(\alpha)\Big)-(n+2)q(\alpha)^2 \;\geq\; 0 \tag{3}$$

for any root α of f .

The inequality (3) can be applied successfully for finding upper bounds for the roots of orthogonal polynomials.

Example

The Legendre polynomial P_n satisfies the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y \;=\; 0\,.$$

From (2), $La(n) = (n-1)\sqrt{\dfrac{n+2}{n(n^2+2)}}$ is a bound for the roots of P_n .

We have thus the following bounds for the largest zeros of P_n:

n	La(P)	LPR
5	0.91084	0.90617
8	0.96334	0.96028
11	0.98021	0.97822
15	0.98922	0.98799
55	0.99917	0.99906
100	0.99975	0.99971

Example

The Hermite polynomial H_n satisfies the differential equation

$$y'' - 2xy' + 2ny = 0 .$$

From (2), $He(n) = (n-1)\sqrt{\dfrac{2}{n+2}}$ is a bound for the roots of H_n . We have thus the following bounds for the largest zeros of H_n:

n	He(P)	LPR
3	1.264	1.224
8	3.130	2.930
12	4.156	3.889
20	5.728	5.387
50	9.609	9.182

A Bound for Hermite Polynomials

Theorem 9 *Let $f \in \mathbb{R}[X]$ be a polynomial of degree $n \geq 2$ that satisfies the second order differential equation*

$$p(x)\, y'' + q(x)\, y' + r(x)\, y \;=\; 0\,, \tag{4}$$

with p, q and r univariate polynomials with real coefficients, $p(x) \neq 0$.

If all the roots of f are simple and real we have

$$8(n-3)q_2(\alpha)^2 + 9(n-2)q(\alpha)q_3(\alpha) \;\geq\; 0\,,$$

where

$$\begin{aligned} q_2 &= q^2 + p'q - pq' - pr\,, \\ q_3 &= (2p'+q)\left(-q^2 - p'q + pq' - pr\right) - pq\left(p'' + 2q' + r\right) \\ &\quad -p^2\left(q'' + 2r'\right)\,. \end{aligned}$$

for any root α of f.

New Upper Bounds for Zeros of Hermite Polynomials

Proposition 10 *The number*

$$\sqrt{\frac{2n^2+n+6+\sqrt{(2n^2+n+6+32(n+6)(n^3-5n^2+7n-3)}}{4(n+6)}}$$

is an upper bound for the positive roots of H_n .

Applications

We consider

$$
\begin{aligned}
He(H_n) &= (n-1)\sqrt{\frac{2}{n+2}}\,, \\
Se(H_n) &= \sqrt{\frac{2n^2+n+6+\sqrt{(2n^2+n+6+32(n+6)(n^3-5n^2+7n-3)}}{4(n+6)}}
\end{aligned}
$$

and obtain

Applications (contd.)

n	He(H_n)	Se(H_n)	LPR
3	1.264	1.224	1.224
8	3.130	2.995	2.930
12	4.156	4.005	3.889
16	4.999	4.844	4.688
20	5.728	5.574	5.387
25	6.531	6.382	6.164
50	9.609	9.484	9.182
60	10.596	10.478	10.159
100	13.862	13.765	13.406
120	15.236	15.146	14.776
150	17.091	17.009	16.629
200	19.801	19.729	19.339

Comparisons with Other Bounds

Several known bounds for the largest positive roots of Hermite polynomials:

$$Bott(H_n) = \sqrt{2n - 2\sqrt[3]{\frac{n}{3}}} \qquad \text{O. Bottema [4]}$$

$$Venn(H_n) = \sqrt{2(n+1) - 2(5/4)^{2/3}(n+1)^{1/3}} \qquad \text{S. C. Van Venn [22]}$$

$$Kras(H_n) = \sqrt{2n-2} \qquad \text{I. Krasikov [10]}$$

$$FoKr(H_n) = \sqrt{\frac{4n - 3n^{1/3} - 1}{2}} \qquad \text{W. H. Foster–I. Krasikov [7]}$$

Comparisons with Other Bounds (contd.)

Comparing the previous bounds with our results we obtain

n	Bott	Venn	Kras	FoKr	He	Se	LPR
4	2.408	2.455	2.449	2.262	1.732	1.659	1.650
16	5.339	5.294	5.477	5.265	4.999	4.844	4.688
24	6.633	6.573	6.782	6.570	6.379	6.228	6.015
64	11.065	10.984	11.224	11.022	10.966	10.851	10.526
100	13.912	13.827	14.071	13.875	13.862	13.765	13.406
120	15.269	15.182	15.422	15.234	15.236	15.146	14.776

The bound $Se(H_n)$ gives the best estimates.

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Thank You Very Much for Your Attention !