The Turbo Principle in Mobile Communications

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Abstract

This overview talk shows that the so-called turbo codes(decoders) entail a much broader principle. It discusses how the feedback of extrinsic information which we call the turbo principle can be used in many mobile communications receivers to improve performance through iterative processing. As an analysis and design tool the EXIT charts of mutual information transfer are used.

1. Introduction

The 'Turbo-Principle' is a general principle in decoding and detection and can be applied to many detection/decoding problems such as serial concatenation, equalization, coded modulation, multiuser detection, multiple-input/multiple-output (MIMO) detection, joint source and channel decoding, low density parity check (LDPC) decoding and others. In almost all cases we can describe the system as a serial concatenation as shown for a few examples in Fig.1. For example in a simple mobile system the inner 'encoder' would be the multipath channel
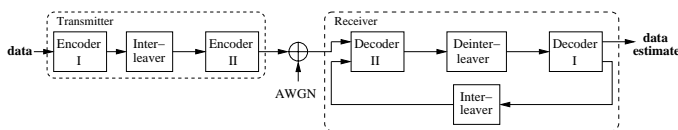
configuration	en-/decoder I (outer code)	en-/decoder II (inner code)
serial code concat.	FEC en-/decoder	FEC en-/decoder
turbo equalization	FEC en-/decoder	Multipath channel/detector
turbo BiCM	FEC en-/decoder	Mapper/demapper
turbo MIMO	FEC en-/decoder	Mapper & MIMO detector
turbo source-channel	source encoder	FEC en-/decoder
LDPC code/decoder	check nodes	variable nodes

Figure 1: A serial concatenated system with iterative detection/decoding.
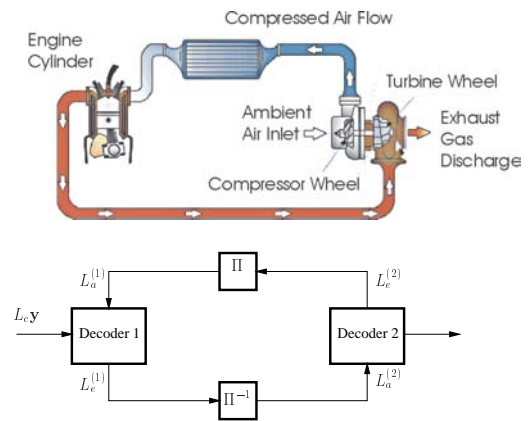
Figure 2: A mechanical turbo engine and a turbo decoder.

The crucial point at the receiver is that the two detectors/decoders are soft-in/soft-out decoders that accept and deliver probabilities or soft values and the extrinsic part of the soft-output of one decoder is passed on to the other decoder to be used as a priori input. This principle has been applied for the first time to decoding of two dimensional product-like codes [1] using similar ideas as in [3] and [4]. Berrou's application is sometimes called a 'turbo code'. Strictly speaking there is nothing 'turbo' in the codes. Only the decoder uses a 'turbo' feedback. This is similar as in a mechanical turbo engine which is shown in Fig.2. In the same way as the compressed air is fed back from the compressor to the main engine the extrinsic information is fed back to the other decoder.

2. Log-Likelihood Ratios and the APP Decoders

Let u be in GF(2) with the elements $\{+1, -1\}$, where $+1$ is the 'null' element under the $\oplus$ addition. Then the **log-likelihood ratio (LLR)** or L-value of the binary variable is

$$L(u) = \ln \frac{P(u = +1)}{P(u = -1)} \qquad (1)$$

with the inverse

$$P(u = \pm 1) = \frac{e^{\pm L(u)/2}}{e^{+L(u)/2} + e^{-L(u)/2}}. \quad (2)$$

The sign of $L(u)$ is the hard decision and the magnitude $|L(u)|$ is the reliability of this decision.
We define the **soft bit** $\lambda(u)$ as the expectation of u where we simultaneously view u in GF(2) and as an integer number

$$\begin{aligned}\lambda(u) = \mathrm{E}\{u\} &= (+1) \cdot P(u = +1) + (-1) \cdot P(u = -1) \\ &= \tanh(L(u)/2). \quad (3)\end{aligned}$$

If P(u) is a random variable in the range of (0,1) then $L(u)$ is a r.v. in the range $(-\infty, +\infty)$ and $\lambda(u)$ a r.v. in the range (-1,+1). The GF(2) addition $u_1 \oplus u_2$ of two independent binary random variables transforms into $\mathrm{E}\{u_1 \cdot u_2\} = \mathrm{E}\{u_1\}\mathrm{E}\{u_2\} = \lambda(u_1) \cdot \lambda(u_2)$. Therefore the L value of the sum $L(u_1 \oplus u_2)$ equals

$$2\tanh^{-1}(\tanh(L(u_1)/2)\cdot\tanh(L(u_2)/2)) = L(u_1) \boxplus L(u_2),$$

abbreviated by the **boxplus operation** $\boxplus$.
The a posteriori probability (APP) after transmitting x over a noisy multiplicative fading channel with amplitude a yielding $y = ax + n$ and the Gaussian probability density function

$$p(y|x) = \frac{1}{\sqrt{2\pi\sigma_c^2}} e^{\frac{-(y - ax)^2}{2\sigma_c^2}} \quad (4)$$

is

$$P(x|y) = \frac{p(y|x)P(x)}{p(y)} \quad (5)$$

and the complementary APP LLR equals

$$L_{CH} = L(x|y) = \ln \frac{P(x = +1|y)}{P(x = -1|y)} = L_c \cdot y + L(x). \quad (6)$$

$L(x)$ is the a priori LLR of x and L_c is the channel state information (CSI):

$$L_c = \frac{2a}{\sigma_c^2} = 4aE_s/N_0 \quad (7)$$

The matched filter output y is Gaussian with $\mathcal{N}(m, \sigma_c^2) = \mathcal{N}(\pm 2a, \sigma_c^2)$. Further, the APP LLR L_{CH} is also Gaussian with $\mathcal{N}(\pm \sigma_{CH}^2/2, \sigma_{CH}^2)$ where $\sigma_{CH}^2 = 2aL_c$ and is determined by one parameter. Note, that the matched filter output has a symmetric pdf $p(-y|x = +1) = p(y|x = -1)$ and is a LLR and as all LLR with symmetric distributions satisfies the consistency condition

$$p(-y|x) = e^{-L_c xy} p(y|x). \quad (8)$$

We further note that for statistically independent transmission, as in dual diversity or with a repetition code

$$L(x|y_1, y_2) = L_{c_1} y_1 + L_{c_2} y_2 + L(x). \quad (9)$$
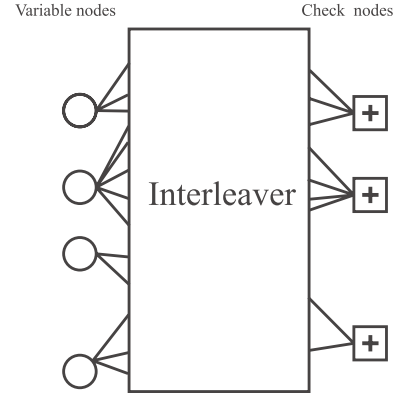

Figure 3: A LDPC code as a serial concatenation of variable nodes and check nodes

Low Density Parity Check Codes and their Decoder

A low density parity check code [6] of rate k/n can be described as a serial concatenation of n variable nodes as inner repetition codes with $n - k$ check nodes as outer single parity check nodes [9]. The Figure 3 shows an irregular LDPC with the i-th variable node (i-th code bit) having $d_{v,i}$ connections via the interleaver to the $n - k$ check nodes where the i-th checks $d_{c,i}$ bits. The same figure can be viewed as the concatenated decoder. The difference to a regular serial concatenated code is that more than one extrinsic message

$$L_{i,j}^{(\mathrm{out})} = L_{c,i} \cdot y_i + \sum_{j=1, j\neq i}^{d_{v,i}} L_{i,j}^{(\mathrm{in})} \quad (10)$$

per code bit x_i, $i = 1...n$ is sent to the outer single parity check (SPC) decoders which return

$$L_{i,j}^{(\mathrm{c,out})} = \sum_{j=1, j\neq i}^{d_{c,i}} \boxplus L_{i,j}^{(\mathrm{c,in})} \quad (11)$$

per check equation $i = 1...n - k$. The decoding result is the overall L value of the inner bits. Note, that algebraic sum in (10) and the boxplus sum in (11) assume statistical independence which after some iterations is not guaranteed.

The APP Decoder

An APP decoder for a linear binary code as shown in Figure 4 code accepts the LLR L_{CH} and the a priori LLR L_A and delivers the extrinsic LLR L_E for all information and/or code bits. Since APP decoding is nonlinear the Gaussian assumption does not hold for L_E. However the following properties are true for all L, be it (L_{CH}, L_A, L_E)
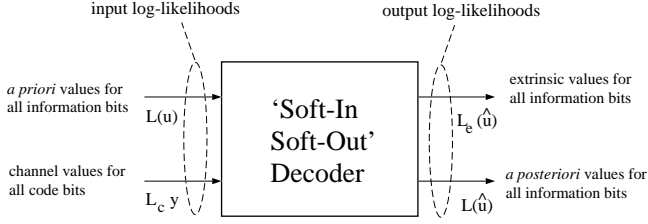
Figure 4: Soft-in/soft-out decoder for turbo iterations

with the respective variances σ^2. They have a symmetric pdf $p(-L|x = +1) = p(L|x = -1)$ and as all LLR with symmetric distributions satisfy the consistency condition $p(-L|x) = e^{-xL}p(L|x)$. They are determined by one parameter because the magnitude of the mean is one half of the variance.

The goal of the soft-in/soft-out algorithm is to provide, as shown in Figure 4, for the given input $\mathbf{y}$ an output for the info bit u_k

$$L(\hat{u}_k) = \ln \frac{P(u_k = +1|\mathbf{y})}{P(u_k = -1|\mathbf{y})} = \ln \frac{\sum\limits_{\substack{(s',s) \\ u_k=+1}} p(s', s, \mathbf{y})}{\sum\limits_{\substack{(s',s) \\ u_k=-1}} p(s', s, \mathbf{y})}. \quad (12)$$

For a binary trellis with state pairs (s', s) one uses the well known APP Bahl-Cocke-Jelinek-Raviv (BCJR) algorithm [2], [5]. Its forward and backward recursion yield

$$\alpha_k(s) = \sum_{s'} \gamma_k(s', s) \cdot \alpha_{k-1}(s'), \quad (13)$$

$$\beta_{k-1}(s') = \sum_s \gamma_k(s', s) \cdot \beta_k(s). \quad (14)$$

The branch transitions, the a priori and the channel metrics (logarithmic probabilities) are given by

$$\gamma_k(s', s) = p(\mathbf{y}_k \mid u_k) \cdot P(u_k), \quad (15)$$

$$P(u_k) = \frac{e^{u_k L(u_k)/2}}{e^{+L(u_k)/2} + e^{-L(u_k)/2}}, \quad (16)$$

$$\ln p(\mathbf{y}_k \mid u_k) = B_k + \frac{1}{2} \sum_{\nu=1}^{n} L_c y_{k,\nu} x_{k,\nu}, \quad (17)$$

for a convolutional code with rate $1/n$ and

$$\ln p(\mathbf{y}_k \mid u_k) = B_{M_k} - \frac{1}{2\sigma^2}|y_k - \sum_{l=0}^{L} x_{k-l} h_l|^2 (18)$$

for a multipath channel with $L + 1$ taps h_l. Several simplifications exist such as the MaxLog approximation [5]. The extrinsic output finally is

$$L_e(\hat{u}_k) = L(\hat{u}_k) - L(u_k). \quad (19)$$

3. Performance Analysis of Soft-in/Soft-out Decoders/Detectors in a Turbo Scheme via EXIT Charts

Many examples of the Turbo principle have shown that the iterative process performs close to the overall maximum likelihood performance of the system although no formal proof is available yet. In many cases the respective channel capacity limit is approached by less than one dB. A great challenge is the analysis of the iterative process which has been attempted via the so-called density evolution analysis [7] and which was successful for low-density parity check codes and for binary erasure channel models. A tool which is especially useful is the **EX**trinsic **I**nformation **T**ransfer (**EXIT**) chart pioneered by Stephan ten Brink in [8], [10] and [13]. Compared to other methods [13] it provides special insight because it measures the mutual information gain in bits at each iteration and each component decoder.

The mutual information between the equally likely x and the respective LLR's L for symmetric and consistent L-values simplifies to

$$I(L; X) = 1 - \int_{-\infty}^{+\infty} p(L|x = +1) \log_2(1 + e^{-L}) dL$$

$$I(L; X) = 1 - \mathrm{E}\{\log_2(1 + e^{-L})\}$$

where the expectation is over the one parameter distribution

$$p(L|x = +1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(L - x\sigma^2/2)^2/2\sigma^2}$$

In case of L_E the expectation is over the measured distribution because after the nonlinear decoder the L-values are not Gaussian any more. However, it can be determined experimentally from the N samples $x_n \cdot L_{E,n}$ which are corrected for positive x by evoking the ergodic assumption

$$1 - \mathrm{E}\{\log_2(1 + e^{-L})\} \approx 1 - \frac{1}{N} \sum_{n=1}^{N} \log_2(1 + e^{-x_n \cdot L_{E,n}})$$

Consequently, we test our respective detectors/decoders with the setup shown in Figure 5.
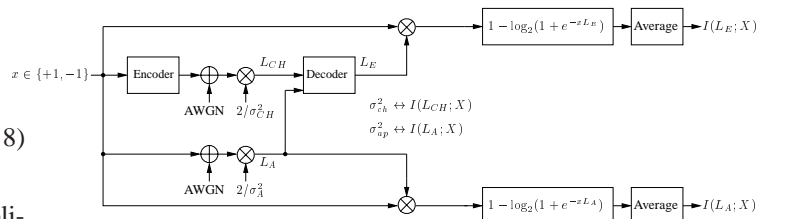

Figure 5: Measurement of the mutual information $I(L_E; X)$.

The extrinsic information transfer function T is measured as

$$I(L_E; X) = T(I(L_A; X)) \qquad (20)$$

The overall assumption for the EXIT chart is a large interleaver to assure statistical independence.

Example for EXIT charts:

We will give an example for a parallel concatenated system with single parity check component codes SPC(n-1,n,2) on the AWGN channel. Assume an (2,3,2) SPC code with a codeword $(+1, +1, +1)$ and decoded with an a priori information at $I(L_A; X) \to 1$.
With the use of the boxplus function

$$L(x_1 \oplus x_2) \quad = \quad L(x_1) \boxplus L(x_2)$$

we obtain for bit $x_1 = x_2 \oplus x_3$ the extrinsic value is $L_E = (L_{CH,2} + L_A) \boxplus L_{CH,3}$ assuming no error. The transfer function eqn.(20) is shown in Figure 6. For the parallel decoder the axes are swapped and the iteration alternates between these curves. The difference $I(L_E; X) - I(L_A; X)$ meaning the difference to the diagonal is the average information gain measured in bits per half iteration. The iteration stops if no information gain is achievable, in the case of such a simple code and low SNR rather early at the point (0.66,0.66). In the case of a good code we will reach the point (1,1) and decode virtually errorfree.
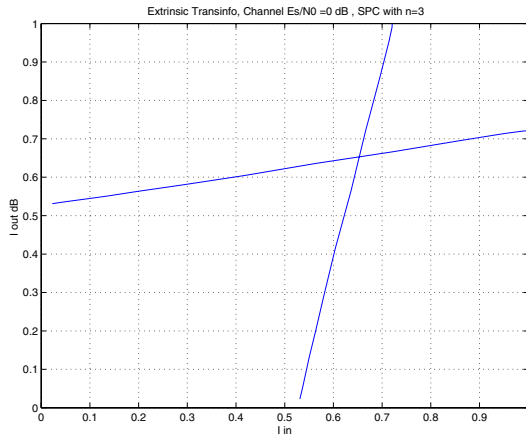

Figure 6: EXIT Chart for a parallel concatenated SPC code.

4. Application Examples in a Mobile Environment

4.1. Coded Equalization of Multipath Channels

A multipath channel can be detected and equalized by the optimal APP detector using the BCJR algorithm with the metric in eqn. (18). However, if the impulse response of the mobile channel is long or the symbol alphabet is large the number of states becomes excessively large. In such a case a soft-in/soft-out linear equalizer can be used which is modified to accept soft decisions. It can work either in the time or in the frequency domain [11], [12]. We will not discuss further this suboptimal but easily realizable equalizer as the inner part of the concatenated system, but show how its EXIT chart can be used to optimize an outer irregular code [14]. The (mirrored) EXIT chart T_{II} of the inner system (5 tap (0.32, 0.63, 0, 0.63, 0.32) ISI channel, BPSK at an SNR of 4 dB) is shown as decoder II in Figure 7.
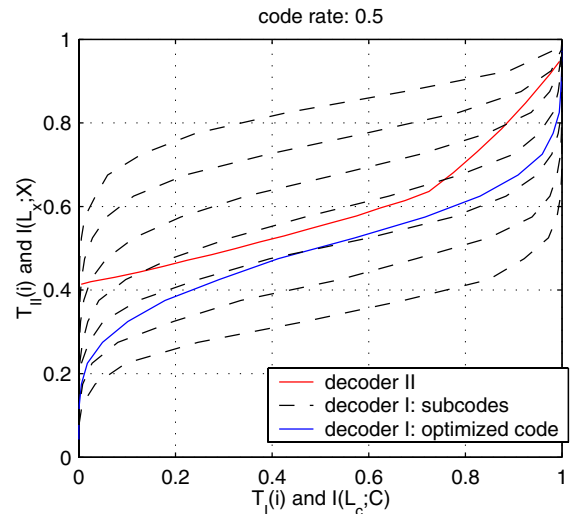

Figure 7: EXIT charts of a coded multipath transmission

For the outer FEC coder we have a family of punctured convolutional codes with memory 4 and $L_R = 7$ rates (4/12, 5/12, 6/12, 7/12, 8/12, 9/12, 10/12). Their EXIT charts are given as dotted lines in Figure 7 and show that a rate 1/2 code cannot achieve convergence in the iterations. However we can construct an irregular code with the same average rate 1/2 as indicated in Figure 8.
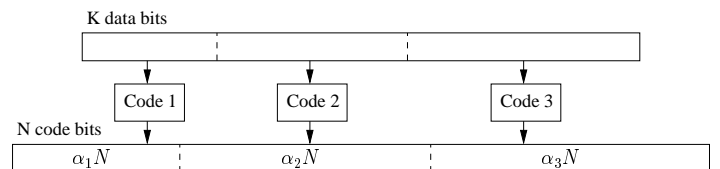

Figure 8: Construction of an irregular code

We have the following constraints on the α_k in this FEC code:

$$\sum_{k=1}^{L_R} \alpha_k = 1,$$

$$\sum_{k=1}^{L_R} \alpha_k R_k = 1/2, \text{ and } \alpha_k \in [0,1], \; k = 1, ..., L_R.$$

Since the L-values of all subcode decoders are symmetric and consistent the overall transfer function is

$$T(i) = \sum_{k=1}^{L_R} \alpha_k T_k(i).$$

By optimizing the set $\{\alpha_i\}$ in such a way that the curves I and II are as close as possible but do not cross in order to keep a reasonable open tunnel for the iterations [14] we obtain the solid curve I in Figure 7.

4.2. MIMO Mobile Channels

Let us assume that we have the following scenario: A multiple-input/multiple-output space time scheme with a MIMO BLAST structure with $N = N_T$ transmit and N_R receive antennas as shown in Figure 9 where the extrinsic output of the detector is send to the outer decoder which in turn supplies a priori values to the MIMO detector. The
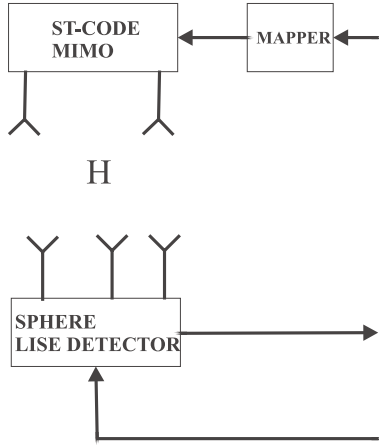

Figure 9: Multiple-input multiple-output (MIMO)channel as inner part of a concatenated coded mobile system with $N = N_T$ transmit and N_R receive antennas.

$N_R \times N$ matrix $\mathbf{H}$ contains the complex channel coefficients $h_k^{(i,j)}$. They are usually assumed to be ergodic, meaning that the mobile channel changes statistically independent after N symbols, as it is the case with frequency hopping. $\mathbf{s}$ is the vector of the symbols transmitted from the N antennas at

block time k. The turbo scheme has as input to the inner block a column bit vector of size $M \cdot N \times 1$

$$\mathbf{x} = (\mathbf{x_1}, ... \mathbf{x_n}, ..., \mathbf{x_N})^{\mathrm{T}} \tag{21}$$

with the M-bit row vectors of size $1 \times N$

$$\mathbf{x_n} = (x_{1,n}, ... x_{m,n}, ..., x_{M,n}), \quad x_{m,n} \in \{+1, -1\}. \tag{22}$$

The constellation mapper maps M bits to one complex signal element $s_n(\mathbf{x_n})$ of the signal vector $\mathbf{s(x)}$ which has size $N \times 1$. After transmission over the complex vector channel we receive a vector of size $N \times 1$ namely $\mathbf{H} \cdot \mathbf{s(x)}$ plus noise

$$\mathbf{y} = \mathbf{H} \cdot \mathbf{s(x)} + \mathbf{n_0} \tag{23}$$

where $\mathbf{y}$ is the vector of the N received symbols. We are interested in the a posteriori LLR of part or all NM bits

$$L(\hat{x}_{m,n}) = L(x_{m,n}|\mathbf{y}) = \ln \frac{\sum_{x_{m,n}=+1} e^{\ln P(\mathbf{x}|\mathbf{y})}}{\sum_{x_{m,n}=-1} e^{\ln P(\mathbf{x}|\mathbf{y})}} \tag{24}$$

A maximum likelihood or symbol APP detector would perform joint detection of all NM bits and maximize

$$L(\hat{x}_{m,n}) = \ln \frac{\sum_{x_{m,n}=+1} e^{-\frac{1}{2\sigma^2} \|\mathbf{y} - \mathbf{Hs(x)}\|^2 + \frac{1}{2} L(\mathbf{x})^{\mathrm{T}} \mathbf{x}}}{\sum_{x_{m,n}=-1} e^{-\frac{1}{2\sigma^2} \|\mathbf{y} - \mathbf{Hs(x)}\|^2 + \frac{1}{2} L(\mathbf{x})^{\mathrm{T}} \mathbf{x}}} \tag{25}$$

if we take the full lengths of all sequences. $L(\mathbf{x})$ are the a priori values of all bits from the turbo feedback of the outer decoder. Evaluating eqn.(25) for all 2^{MN} possible data is prohibitively complex even for moderate M and N. A possible way is to use a reduced complexity sphere detector proposed in [16] which finds a candidate list of transmit vectors $\mathbf{x}$ and evaluate (25) only for those candidates. To find the candidates we compute the center (zero forcing) solution with the upper index $^{\mathrm{H}}$ denoting the Hermitian (Conjugate complex)

$$\hat{\mathbf{s}}(\mathbf{y}) = \mathbf{H}^{\mathrm{H}} (\mathbf{H}^{\mathrm{H}} \mathbf{H})^{-1} \mathbf{y} \tag{26}$$

of a search sphere which would be equal to the transmitted vector $\mathbf{s}$ in case of noiseless transmission. Applying e.g. a Cholesky factorization we obtain a lower triangular matrix $\mathcal{L}$ which satisfies

$$\mathcal{L}^{\mathrm{H}} \mathcal{L} = \mathbf{H}^{\mathrm{H}} \mathbf{H}. \tag{27}$$

Since we have now a lower triangular matrix $\mathcal{L}$ using (26) and (27) we can efficiently evaluate the first part in the normalized metric in the exponent of (25) after dropping the parts which do not depend on $\mathbf{x}$

$$\frac{1}{2\sigma^2} \| \mathbf{H}(\mathbf{s(x)} - \hat{\mathbf{s}}(\mathbf{y})) \|^2 = \frac{1}{2\sigma^2} \| \mathcal{L}(\mathbf{s(x)} - \hat{\mathbf{s}}(\mathbf{y})) \|^2 \tag{28}$$

In [16] the MaxLog approximation and a geometric interpretation is used to find the candidates for the list around

the zero forcing solution. However, since the metric after the Cholesky factorization is now additive we can apply a modified sequential search on a tree using the stack algorithm [17]. The size of the stack controls the performance of MIMO detector. If it is large enough it achieves APP performance. Even if some of the pathes of the stack do not reach the full length we can use an augmented stack to utilize all available information in order to obtain the best soft output. As shown in [17] considerable complexity reductions can be achieved.

A similar Turbo detection as in the MIMO channel can be performed in multiuser detection as treated in [15].

5. Conclusions

We have shown in a wide range of examples for mobile applications how the turbo principle can be used for iterative detectors and decoders. We have not treated here the parameter estimation of the varying channel which can be also integrated in the turbo process effectively transforming data bits into training bits. In such a way the channel estimation improves with each iteration.

Acknowledgement

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