Course "Polynomials: Their Power and How to Use Them",
JASS'07

Differential Polynomials

Stephan Ritscher

Fakultät für Informatik
TU München

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Differential Polynomial Ideals - TOC

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Differential Ring

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Θ denotes the free abelian monoid generated by Δ, alias Δ^*, members of Θ are called *derivations*.

Differential Ideal

Definition 2 (Differential Ideal)

An *differential ideal* I is a ideal of R with $\forall \delta \in \Delta : \delta I \subset I$. We write:
$[S]$ differential ideal generated by set S
$\{S\}$ perfect differential ideal generated by set S

Example

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6. ...

Notation

$F\{X\}$	differential polynomials over field F with variables X
$\mathrm{lm}(f)$	leading monomial of f
$\mathrm{lc}(f)$	leading coefficient of f
$\mathrm{lt}(f)$	$= \mathrm{lc}(f)\mathrm{lm}(f)$ leading term of f
$\theta x = x_{(\theta)}$	derivation $\theta \in \Theta$ of variable x
$\mathrm{ord}(\delta^\alpha x)$	$= \sum_{i=1}^{n} \alpha_i$ order of $\delta^\alpha x$
$\deg(v^\beta)$	$= \sum_{i=1}^{r} \beta_i$ degree of v^β, $\delta^\alpha \in \Theta$
$\mathrm{wt}(v^\beta)$	$= \sum_{i=1}^{r} \beta_i \mathrm{ord}(v_i)$ weight of v^β

Nonrecursive Ideals

Example 4

Consider over $\mathbb{Z}\{x\}$ with $\Delta = \{d\}$:
$f_i = (d^i x)^2$ for $i \geq 0$ and $I_k = [f_0, \dots, f_k]$.
Then: $I_0 \subsetneq I_1 \subsetneq \dots$

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- $\alpha_{i,j} = 0$ for $j \neq 2n - 2i$ ($\mathrm{wt}(f_n) = 2n$, $d^j f_i$ are isobaric).

Nonrecursive Ideals (2)

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Proof.

$$\Rightarrow \quad f_n = c_0 d^{2n} f_0 + c_1 d^{(2n-2)} f_1 + \ldots + c_{n-1} d^2 f_{n-1} \text{ for } c_i \in \mathbb{Z}$$

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- Analogous reasoning for $d^{2n-2i} f_i$ yields $c_i = 0$
- Contradiction: $f_n \neq 0$

□

Nonrecursive Ideals (3)

Example 6

Let $S \subset \mathbb{N}_0$ and $I_S = [\{f_i : i \in S\}]$. Then

$$f_i \in I_S \Leftrightarrow i \in S$$

So for a nonrecursive set $S \subset \mathbb{N}_0$ there is no algorithm to decide if a given differential polynomial g is in I_S.

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3. $f < g \Rightarrow hf < hg$ for all $f, g, h \in M$

Examples of Rankings

$X = \{x_1, \ldots, x_n\}$ with $x_1 < \ldots < x_n$.

Example 9 (Lexicographic Ranking on ΘX)

Consider a monomial ordering $<$ on the differential operators Θ. Then the *lexicographic ranking* is given by $\theta x_i < \eta x_k$ iff $i < k$ or $i = k$ and $\theta < \eta$.

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- For $|X| = |\Delta| = 1$ there is only one ranking: $x_{(i)} < x_{(i+1)}$

Examples of Admissible Orderings

$f = \prod_{i=1}^{r} v_i^{\alpha_i}$ with $v_1 > \ldots > v_r$.
$g = \prod_{i=1}^{s} w_i^{\beta_i}$ with $w_1 > \ldots > w_s$.

Example 11 (Lexicographic Ordering on M)

Given an ranking on ΘX.
$f <_{lex} g$ iff $\exists k \leq r, s : v_i = w_i$ for $i < k$ and $v_k < w_k$ or $v_k = w_k$ and $\alpha_i < \beta_i$ or $v_i = w_i$ for $i \leq r$ and $r < s$.

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Example 12 (Graded (by Degree) Reverse Lexicographic Ordering on M)

Given an ranking on ΘX.
$f <_{degrevlex} g$ iff $\deg(f) < \deg(g)$ or $\deg(f) = \deg(g)$ and $f <_{revlex} g$.

Reduction

Definition 13

f is *reduced* by g to h iff $\exists\theta \in \Theta, m \in M$ such that $\mathrm{lm}(f) = \mathrm{lm}(m\theta g)$ and $h = f - \frac{\mathrm{lc}(f)}{\mathrm{lc}(g)} m\theta g$.

f is *reducable* by g, iff there is an h such that f is reduced by g to h.

Reduction Algorithm

```
Procedure:  Reduce(f, g)
if (deg(lm(f)) < deg(lm(g)) || wt(lm(f)) < wt(lm(g)))
  return f;
if (lm(g)|lm(f))
  return Reduce(f − (lt(f)/lt(g)) ∗ g, g);
for (i = 1; i <= m; i++) {
  t = Reduce(f, δ_i g);
  if (t != f) return Reduce(t, g);
}
return f;
```

Monoideals and Standard Bases

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Definition 15 (Standard Basis)

$G \subset I$ is called a *standard basis* iff $\operatorname{Im}(G)$ generates $\operatorname{Im}(I)$ as monoideal.

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Example 16 (Monoideal - Lexicographic Ordering)

Members of the monoideal I generated by x^2 over $F\{x\}$ with $\Delta = \{d\}$ using lexicographic ordering ($x_{(k)} := d^k x$):

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4. $mx_{(k)}x$
5. BUT $(x_{(k)})^r \notin I$

Examples (2)

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Members of the monoideal I generated by x^2 over $F\{x\}$ with $\Delta = \{'\}$ using graded lexicographic ordering ($x_{(k)} := d^k x$):

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7. $mx_{(k)}^2$

Membership Problem

Theorem 18

Let G be a set of polynomials, I a differential ideal. Then the following propositions are equivalent:

Proof:

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1. *G is a standard basis of I.*

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Let G be a set of polynomials, I a differential ideal. Then the following propositions are equivalent:

1. *G is a standard basis of I.*
2. *For $f \in F\{X\}$ yields: $f \in I \Leftrightarrow f$ is reduced to 0 by G.*

Proof:

Membership Problem

Theorem 18

Let G be a set of polynomials, I a differential ideal. Then the following propositions are equivalent:

1. *G is a standard basis of I.*
2. *For $f \in F\{X\}$ yields: $f \in I \Leftrightarrow f$ is reduced to 0 by G.*

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$\Leftarrow$ $g \in G \Rightarrow g$ is reduced to 0 by $G \Rightarrow G \subset I$
$f \in I \Rightarrow f$ is reduced to 0 by $G \Rightarrow \mathrm{lm} I \subset \mathrm{lm}(M \Theta G)$

Infinite Standard Bases

Example 19

Remember $I = [x^2]$ over $F\{x\}$ with $\Delta = \{d\}$. Then for every $r \geq 0$ there is an $q > 1$ such that $(x_{(r)})^q \in I$.

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  Therefore $(x_{(r)})^s$ for every $r \geq 0$ for some $s > 0$ is in every standard basis ($\rightarrow$ infinite).
- DEGREVLEX: x^2 is a standard basis.

Infinite Standard Bases (2)

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Conjecture: There is no finite standard basis for $[x_{(1)}x]$ for no monomial ordering.

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belong to the ideal $[x_{(1)}x]$.

Manifolds

We choose e.g. F as set of all meromorphic functions.

Definition 22

Let Σ be a system of differential polynomials over $F\{x_1, \ldots, x_n\}$, F_1 an extension of F.
If $Y = (y_1, \ldots, y_n) \in F_1^n$ such that for all $f \in \Sigma$ $f(y_1, \ldots, y_n) = 0$, then Y is a *zero* of Σ. The set of all zeros of Σ (for all possible extentions of F) is called *manifold*.

Unions of Manifolds

- Let M_1, M_2 be the manifolds of Σ_1, Σ_2. If $M_1 \cap M_2 \neq \emptyset$ then $M_1 \cap M_2$ is the manifold of $\Sigma_1 + \Sigma_2$. $M_1 \cup M_2$ is the manifold of $\{AB : A \in \Sigma_1, B \in \Sigma_2\}$.

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- M is called *reducible* if it is union of two manifolds $M_1, M_2 \neq M$.
- Otherwise it is called *irreducible*.

Irreducible Manifolds

Lemma 23

M is irreducible $\Leftrightarrow$
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$\Rightarrow$ Assume $\exists A, B$ such that AB vanishes over M, but A, B don't. Then the manifolds of $\Sigma + A$, $\Sigma + B$ are proper parts of M, their union is M.

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- $\Leftarrow$ Let M be proper union of M_1, M_2 with systems Σ_1, Σ_2. Then $\exists A_i \in \Sigma_i$ be differential polynomials that do not vanish over M. $A_1 A_2$ vanishes over M.

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- I.e. irreducible manifolds correspond to prime ideals.

Decomposition

Theorem 24

Every manifold is the union of a finite number of irreducible manifolds.

Decomposition (2)

Consider differential polynomials over $F\{x\}$ with $\Delta = \{d\}$ and F the meromorphic functions:

Example 25

Let $\Sigma = [f]$ with $f = x_{(1)}^2 - 4x$. Then $df = 2x_{(1)}(x_{(2)} - 2)$.

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- $x_{(2)} - 2 = 0$ has the solution $x(t) = (t + b)^2 + c$. Again $c = 0$.
- There are no other solutions.

The Theorem of Zeros

Theorem 26

Let $\Sigma = [f_1, \ldots, f_k]$ *with manifold* M*. If* g *vanishes over* M *then* $g^s \in \Sigma$ *for some* $s \in \mathbb{N}_0$*.*

The Theorem of Zeros

Theorem 26

Let $\Sigma = [f_1, \ldots, f_k]$ with manifold M. If g vanishes over M then $g^s \in \Sigma$ for some $s \in \mathbb{N}_0$.

- So the manifolds are represented by perfect ideals.

The Ritt-Braudenbush Theorem

Theorem 27

Every perfect differential ideal has a finite basis.

Membership Test for Perfect Differential Ideals/Manifolds

Let Σ be a finite system of differential polynomials.
Question: Is $f \in \{\Sigma\}$?

- Resolve Σ into prime ideals (resp. the corresponding manifold into irreducible manifolds).

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- Test if the remainder of f with respect to the characteristic sets of the prime ideals is zero.

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Conclusion

- Differential polynomials can be used to model differential equation systems.
- There are many problems in contrast to algebraic polynomials (no finite (standard) bases, monomial ideals depend on ordering).
- Manifolds (solutions) correspond to perfect ideals, that are easier to handle.
- For some important problems (finite) algorithms exist.

Thank you for the attention

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