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Dynamic inverse problem in acoustic media

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Abstract

Solution of inverse problem is one of the most important questions in many fields of science and knowledge. Only a small range of simple inverse problems could be solved by mathematical means. In geophysics and other sciences there exist a number of kinematic and dynamic inverse methods which inherently assume preliminary knowledge of the reference medium. In order to construct the regular reference medium the author suggests Gelfand-Levitan-Moses method. The method is based on mathematical theory and allows reconstructing the solution of inverse dynamic problem for a simplest models. The model parameters are estimated from the recorded dataset. As far as such an estimation of a reference model has fully mathematical basis it is automatically more regular than empirical. Numerical examples for 2D case illustrate the potential of the method and give a direction for further investigations. The method could be used in applied geophysics or may have different applications in fields where one must deal with inverse problems.

Introduction

An inverse problem is the task that often occurs in many branches of science and mathematics where the values of some model parameters must be obtained from the observed data. The transformation from data to model parameters is a result of the interaction of a physical system, e.g., the Earth, the atmosphere, gravity etc. Inverse problems arise for example in geophysics, medical imaging (such as computed axial tomography and eeg), remote sensing, ocean acoustic tomography, nondestructive testing, and astronomy. Inverse problems are typically ill posed, as opposed to the well-posed problems more typical when modeling physical situations where the model parameters or material properties are known. Of the three conditions for a well-posed problem suggested by Jacques Hadamard (existence, uniqueness, stability of the solution or solutions) the condition of stability is most often violated.

In the sense of functional analysis, the inverse problem is represented by a mapping between metric spaces. While inverse problems are often formulated in infinite dimensional spaces, limitations to a finite number of measurements, and the practical consideration of recovering only a finite number of unknown parameters, may lead to the problems being recast in discrete form. In this case the inverse problem will typically be ill-conditioned. In these cases, regularization may be used to introduce mild assumptions on the solution and prevent overfitting.

As far as inverse problem is very complicated to solve analytically, there are a lot of developed practical techniques based on matching of supposed parameters values with the given data. In such kind of investigations the initial guess plays a big role.

Basically inverse problems can be divided into kinematic and dynamic problems. Kinematic problems deal only with propagation time of a signal through a medium while dynamic problems involve full registered wavefield.

Inverse problems in geophysics

In geophysics inverse problem can be stated as follows: given the registered seismic data at the medium surface (or in a well) to reconstruct the medium parameters:

P-wave velocity, V_p[1]
S-wave velocity, V_s[2]
density, ρ
etc.

Kinematic methods like Kirchhoff migration[3] technique can reconstruct well the seismic horizons - the boundaries between layers of different parameters. Dynamic methods like full waveform tomography[4] allows reconstructing the absolute values between layers as well as horizons. But both methods are similar in a sense that they need a proper starting model, the reference medium which is the initial guess for the algorithm.

Formulation of dynamic inverse problem in acoustic media

In order to make the right initial guess one can try to estimate model by analytical means i.e. do it mathematically for a simple averaged model for which a unique solution can be obtained.

First we will consider acoustic media. This means only P-waves are presented. Also, for simplicity we deal with 2D case. Extension to 3D case is trivial.

Acoustic 2D equation:

$$\frac{1}{\rho v^2}U_{tt}(x,z,t) - \frac{\partial}{\partial z}\left(\frac{1}{\rho}U_z(x,z,t)\right) - \frac{\partial}{\partial x}\left(\frac{1}{\rho}U_x(x,z,t)\right) = 0, \quad (1)$$

where v is acoustic velocity, ρ is density, U_{tt} is a second time derivative of the pressure $U(x,z,t)$, U_x and U_z are spatial derivatives.

As usual for mathematical formulation of problems we should specify zero initial condition:

$$U(x,z,t<0) = 0, \quad (2)$$

which means that there were no any disturbance before. Boundary condition serves here as a source:

$$U(x,z=0,t) = \delta(t)\delta(x), \quad (3)$$

where $\delta(t)$ and $\delta(x)$ are Dirac's delta functions [5] on time and space correspondingly.

To solve the inverse problem we should have the recorded data, and they are:

$$U_z(x,z=0,t) = R(x,t) \quad (4)$$

Finally, the problem is to obtain parameters $v(x,z)$, $\rho(x,z)$ from initial data $R(x,t)$.

The Gelfand-Levitan-Moses method

For the reconstruction of acoustic parameters within a specified problem the Gelfand-Levitan-Moses method is suggested to be used. This method was originally developed in 1951 for the Schrodinger equation by Gelfand, Levitan ([1]) and Kay, Moses ([2]). Blagovestchenskii applied the method to the 1D wave equation ([3]). Also, the method was extended to the 2D and 3D acoustic wave equation. For other theoretical results and first applications to 3D inhomogeneous acoustic media see [4]-[6].

The Gelfand-Levitan-Moses (GLM) method has the remarkable property that existence and uniqueness of the solution of the inverse problem can be proven in case of the following types of sources: $\delta(t), \delta'(t), \delta''(t), ...$

[1] Propagation velocity of a pressure (acoustic) wave
[2] Propagation velocity of a shear wave
[3] A method of seismic migration that uses the integral form (Kirchhoff equation) of the wave equation.
[4] A technique to measure and display the three-dimensional distribution of velocity or reflectivity of a volume of the Earth by using numerous sources and receivers at the Earth's surface.
[5] A function representing an infinitely sharp peak bounding unit area

Let's assume that velocity $v(x, z)$ depends only on depth, i.e. $v = v(z)$. Exists the theory that allows slight lateral inhomogeneity, but it is out of frames of the present paper.

Let:

$R(t, k_x)$ be the Fourier transform in x of initial dataset $R(t, x)$ for a single value of k_x.
$R_1(t, k_x)$ be equal to $R(t, k_x) - \delta'(t)$.
$R^+(t, k_x)$ be the odd continuation of the function $R_1(t, k_x)$ to a negative time.

Then for the delta function source described above GLM method leads to a following integral equation with real parameter y:

$$\Theta(t, y) - \frac{1}{2}\int_{-y}^{y} d\tau\Theta(\tau, y)\int_{\tau}^{y} d\eta R^+(t-\eta, k_x) = \frac{1}{2}\int_{-y}^{y} d\tau R^+(t-\tau, k_x) \tag{5}$$

One can obtain velocity $v(z)$ for an arbitrary value k_x using the solution $\Theta(t, y)$ and the following equations:

$$\frac{\left(\sqrt{\sigma(y)}\right)''}{\sqrt{\sigma(y)}} + k_x^2\frac{1}{\sigma^2(y)} = -2(\lim_{t\to y}\Theta(t, y))'_y, \tag{6}$$

$$v(y) = \frac{1}{\sigma(y)}, \tag{7}$$

$$z = \int_0^y v(y)dy. \tag{8}$$

To solve ordinary differential equation (6) one must know $v(0)$ and $v'(0)$ which in fact are always known, especially for marine acquisition. Equation 5 is an integral equation of a second kind and can be solved be the means of linear algebra.

As far as delta function source could not be represented numerically, the following way is suggested. Instead of delta function one can use Ricker wavelet[6] as a signal form. Ricker wavelet is usually used in seismic modelling. The waveform of Ricker wavelet can be used in a Wiener filter for the recorded data. After filtering the dataset transforms to one that corresponds to a pulse source, i.e. delta function. This means that deconvolution of the data with known waveform will give the necessary result.

Besides, another method to get the right response is double integration of the data since Gaussian function is a simplest way to mimic delta function source.

Numerical results

Numerical examples for 2D case illustrate the potential of the method and give a direction for further investigation.

The simple marine acquisition scheme and velocity model of a layered half-space is shown on fig.1. A set of seismic receivers with 12m spacing is situated on 4m depth. Air gun[7] is placed 10m deeper than receivers profile. Different layers has individual color that mean different medium properties.

For such kind of model the GLM method could be easily implemented. Having recorded the seismograms during the survey, one must make a Fourier transform of the data in space domain over horizontal direction of a profile. It is done numerically in the following way:

$$R(t, k_x) = \int R(t, x)\cos k_x x dx \approx \Delta x\sum_{n=1}^{N} R(t, x_n)\cos(k_x x), \tag{9}$$

where x_n are positions of receivers from 1 to N.

Figure 2 represents layered half-space used in first numerical example. Vertical velocity variations is better seen on fig.3.

Forward modelling was computed by the finite-differences code based on Fourier method (see [9]). Seismograms for Ricker signal with prevailing frequency of 30 Hz are shown on fig.4. Multiple reflections

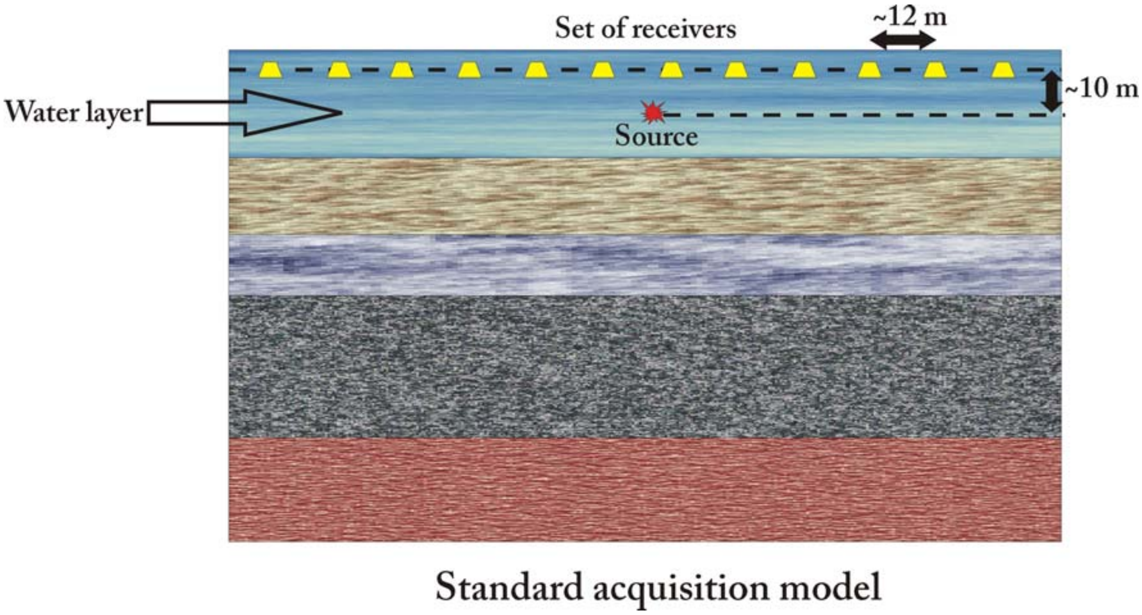

Figure 1: Layered half-space with water surface

from sea bottom and sea surface are clearly seen on the data. These multiples are harmful for kinematic methods but they are fully used in dynamic approach since it involves the full wavefield.

The result of Fourier transform for the data is shown on fig.5. This dataset is initial for the algorithm described above.

Processing of the dataset from fig.5 with a special code written by the author leads to a result represented in fig.6. Reconstructed velocity distribution (red) represents the real profile well if the spatial extent of the variations is bigger than the wavelength. For smaller variations the resolution of the method is not enough. It was shown previously ([7],[8]) that for smooth velocity models method provides better results.

Correct reconstruction of a velocity model is possible up to the depth that corresponds to the time moment when direct wave from source reaches the last receiver of the receiver set. This fact is in good agreement with fig.6 and other results for different velocity models (figures 7,8).

Result for another velocity distribution is represented in fig.7. In this case there is a sharp peak at the 125m depth which couldn't be properly imaged. The spatial extent of the peak is too small. Whereas slow variation is properly imaged. Again, the depth of 550m is critical in this case.

Another result is shown on fig.8. Large steps on shallow level is imaged well while following small steps are not reconstructed.

One of the main problems encountered during investigations is noise. Preliminary tests showed that low noise affects the results dramatically. Other interesting results could be found in [7],[8].

[6]Second derivative of Gaussian function. Generally known as 'Mexican hat'.

[7]A source of seismic energy used in acquisition of marine seismic data. This gun releases highly compressed air into water.

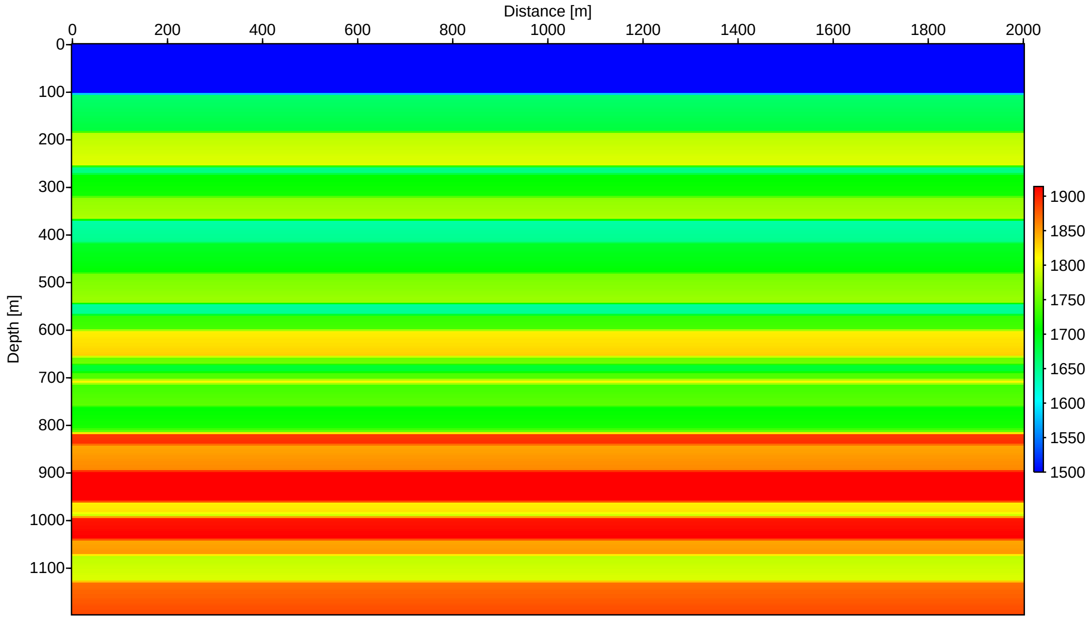

Figure 2: Layered half-space used in first numerical example. Color corresponds to acoustic velocity value in m/s shown on the right.

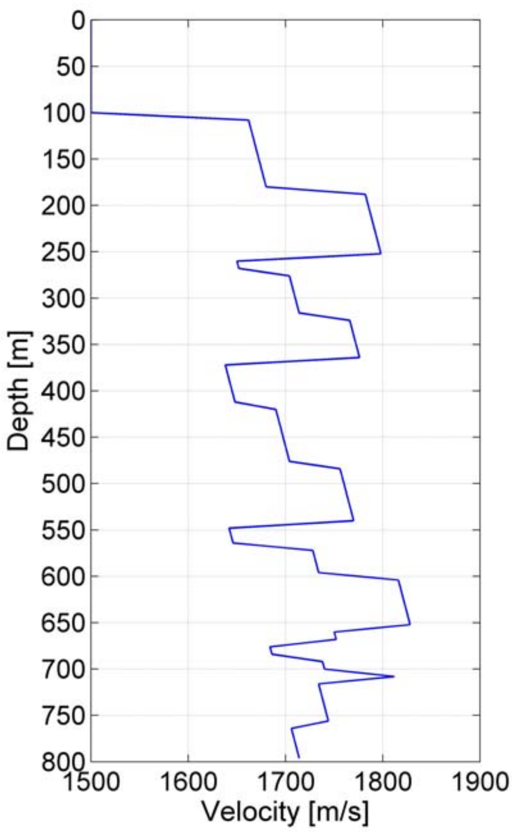

Figure 3: Vertical velocity profile of the layered half-space used in first example.

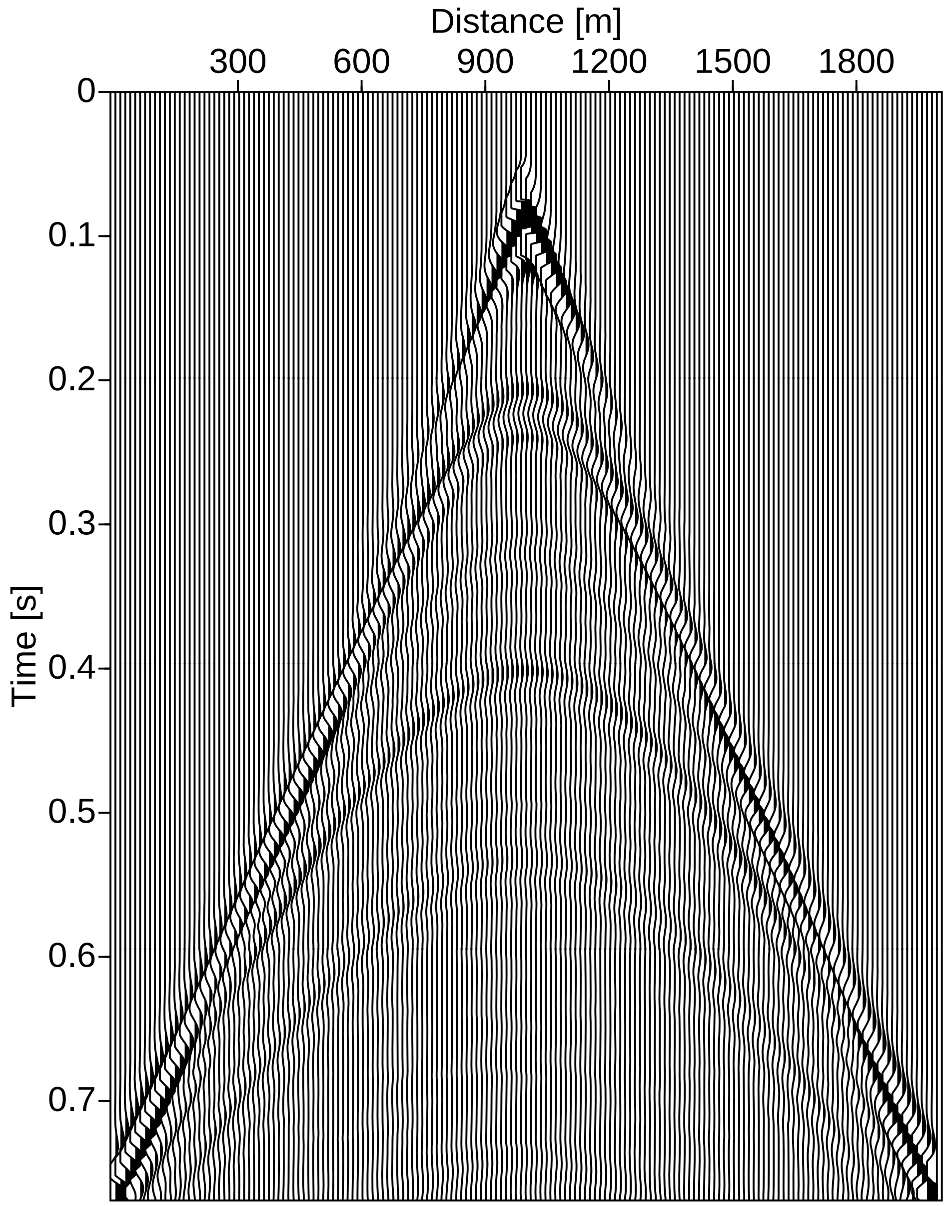

Figure 4: Seismograms for the first numerical example.

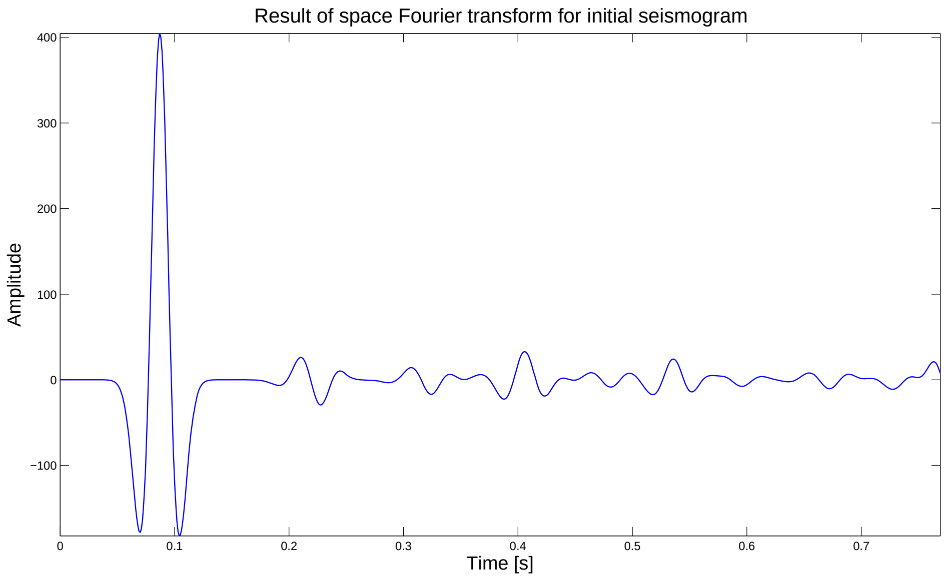

Figure 5: Initial dataset for the GLM algorithm

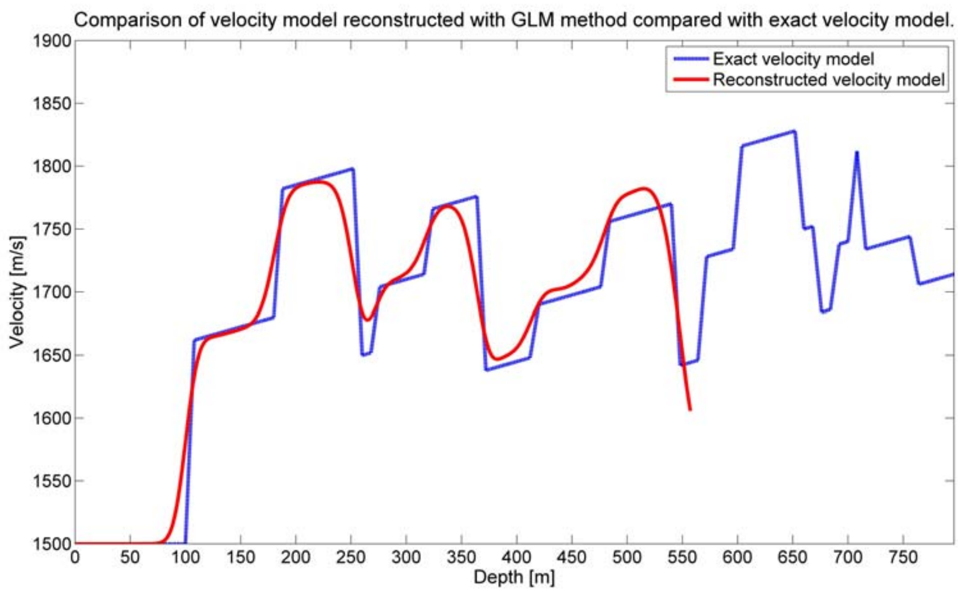

Figure 6:

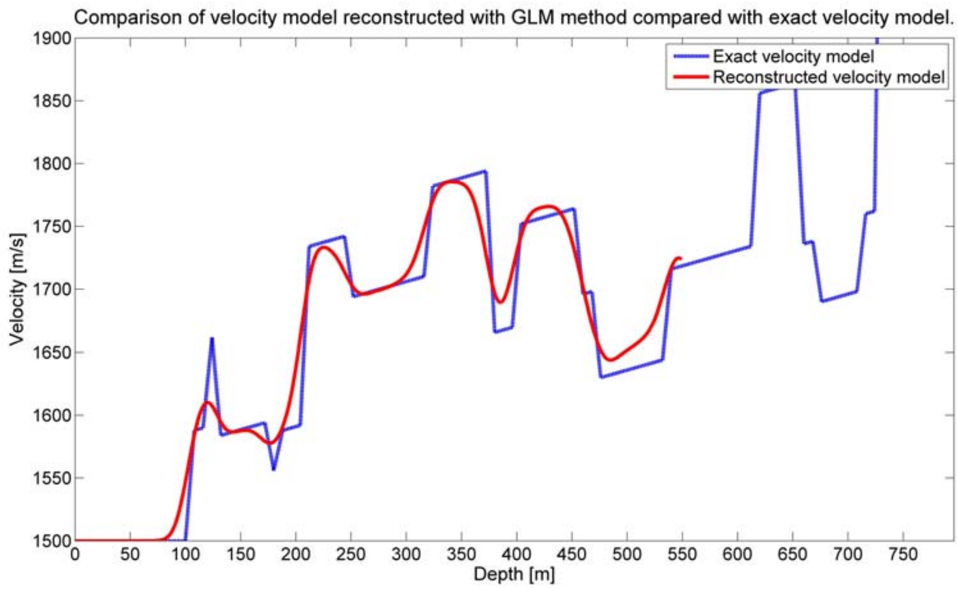

Figure 7:

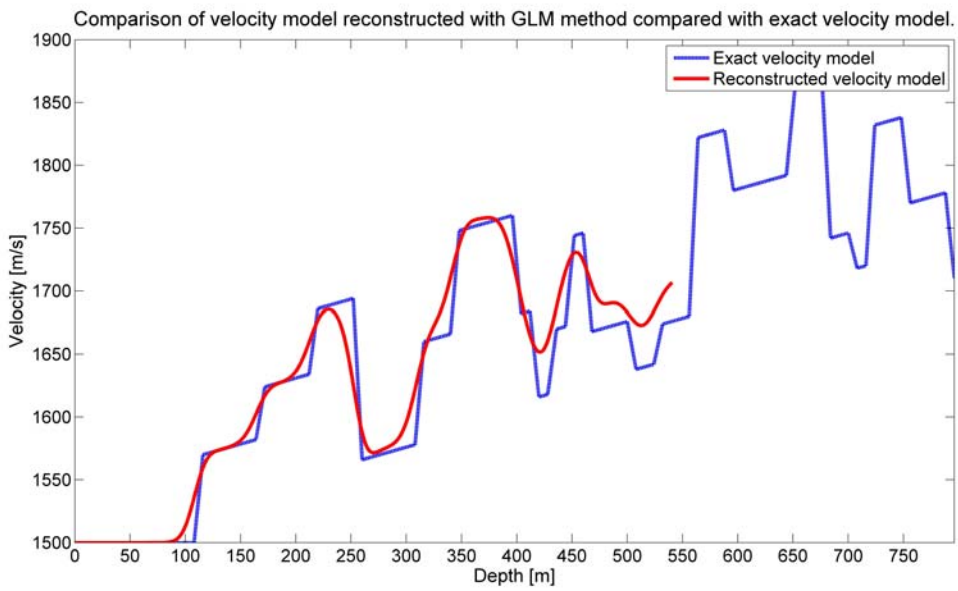

Figure 8:

Conclusions

- The GLM method was introduced for reconstruction of a solution of a dynamic inverse problem for 2D acoustic laterally homogeneous media.
- The method is proved to be efficient and accurate for smooth velocity distributions.
- Main disadvantage is low stability in presence of noise.
- Current results lead to a conclusion that suggested method can be used for construction of preliminary reference medium which is important for other inverse methods used in geophysics and other fields of science.

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