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Fundamental Algorithms

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1. Landau Symbols

The definitions above offer us a very detailed look at an algorithm's resource usage. But as we will see, we will sometimes want to reduce the amount of detail.

- Uniform and logarithmic time and space complexities depend considerably on the machine model, i.e. the type of computer the algorithm is implemented and run on. Differences between machine models include the instruction set, the cost of individual instructions, etc.

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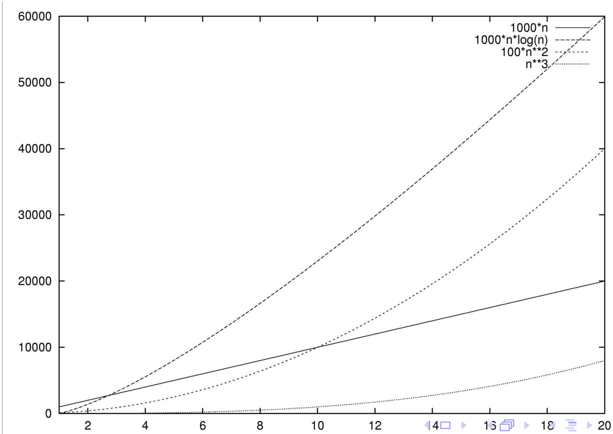
- Comparing the efficiency of algorithms by the functions $t(n)$ and $s(n)$ describing the upper bounds of time and space usage makes little sense for small values of n.
  One function $t_1(n)$ that clearly dominates another function $t_2(n)$ may yield smaller values for small input sizes n.
  Example: Remember our the functions seen in the first lecture.



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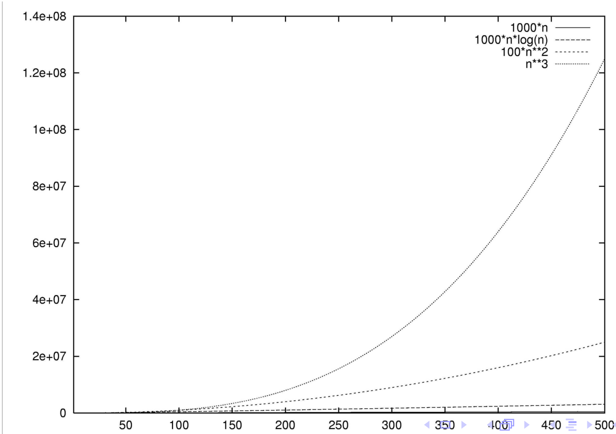

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- $\Omega(f) := \{g : \mathbf{N} \longrightarrow \mathbf{R}^+ : (\exists c \in \mathbf{R}^+, n_0 \in \mathbf{N} : \forall n \geq n_0 : g(n) \geq c \cdot f(n))\}$
- $\Theta(f) := O(f) \cap \Omega(f)$
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Some intuition on Landau symbols:

- $f = O(g)$ means that, for almost all n, $g(n) \leq c \cdot f(n)$, where c is some existing positive constant. In other words, f grows at most as fast as g as n grows.
- $f = \Omega(g)$ conversely means that f grows at least as fast as g asymptotically.
- $f = \Theta(g)$ means that, up to some constant factor, f and g grow equally fast asymptotically.
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Example 2

- $n^2 + 2n - 1 = O(n^2)$: select $n_0 = 1, c = 2$
- $n^2 + 2n - 1 = \Omega(n^2)$: select $n_0 = 1, c = 1$
- Hence: $n^2 + 2n - 1 = \Theta(n^2)$
- Equally easy to prove: $3n + 4 = o(n^2)$ and $n^2 + 2n - 1 = \omega(n \log n)$
- $n^2 \log + 4n(\log n)^2 = \Theta(n^2 \log n)$
- $\log n = o\left(\sqrt{n}\right)$ (Try to prove this by yourself)
- General polynomials: $\sum\limits_{i=0}^{k} a_i n^i = \Theta(n^k)$ if $a_k > 0$

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It holds that

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It holds that

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Definition 4

A function f is called

- constant if $f(n) = \Theta(1)$
- logarithmic if $f(n) = O(\log n)$
- linear if $f(n) = O(n)$
- quadratic if $f(n) = O(n^2)$
- polynomial if $f(n) = O(n^k)$ for some $k \in \mathbf{N}$
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Chapter III Sorting Algorithms

1. Introduction

We are given a non-sorted sequence $D_1, D_2, \ldots, D_n$ consisting of n data elements. Each data element D_i consists of a key k_i and some data contents X_i.

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The sorting problem consists in determining a permutation, i.e. a bijection $\pi : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ such that

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2. Algorithm Design by Induction

We now introduce a methodology that allows us to derive first algorithmic approaches to many types of problems in which the sizes of problem instances depend on a parameter n. We are interested in an algorithm that can solve arbitrary problem instances of size n, for all $n \in \mathbf{N}$.

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  - We pretend to have a method that solves problem instances of size $< n$. This is often called the induction hypothesis.
  - For instances of size n, we devise a method that decomposes the given problem instance into smaller instances of size $< n$, e.g. $n - 1$ or $\lfloor n/2 \rfloor$. It is essential that this be instances of the same problem, just smaller.

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   - We use the induction hypothesis to solve the smaller instances. This can be translated into recursive calls.
   - Finally, we give a method to recombine the smaller solutions into a solution of the original size-n problem.

Examples of this will be seen in the follwing sections (and in most of the remainder of this course).

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   - Let us assume we know how to sort $(n-1)$ keys.
   - To reduce the problem size from n down to $(n-1)$, we remove one key from the array. Here we select the smallest key. (Hence "Selection Sort"). We need to make sure that the form of the problem does not change. The remaining keys must be stored in an array, at consecutive positions. To avoid a gap, we exchange the smallest key with the one at the first position.
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Let us suppose that the array can be accessed through a global variable $A[]$.

Algorithm:

```
void SelectionSort_recursive(unsigned $l, r$){
    if $(l = r)$ then return
    else let $k_s := \min\{k_l, k_{l+1}, \ldots, k_r\}$
        swap $A[k_l]$ and $A[k_s]$
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This function is initially called by "SelectionSort_recursive(1,n)".

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For the sake of efficiency, and sometimes code legibility, we often try to avoid recursion. An equivalent algorithm can be formulated using iteration rather than recursive calls. The strategy applied here is only slightly different.

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```
void SelectionSort_iterative(key $A[]$, unsigned $n$){
  for $i := 1$ to $n - 1$ do
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Complexity:

We measure the efficiency of the above algorithm by counting the number $c(n)$ of key comparisons needed to sort n keys. (This is the traditional way of comparing the complexities of sorting algorithms.)

Let us consider the iterative algorithm. In the i-th iteration of the outer for-loop, we have $(n-i)$ comparisons. ($A[i+1]$ vs. $A[i]$, $A[i+2]$ vs. $A[i]$, $\cdots$, $A[n]$ vs. $A[i]$). Hence:

$$c(n)=\sum_{i=1}^{n-1}(n-i)=\sum_{i=1}^{n-1} i=\frac{n(n-1)}{2}=\frac{1}{2}(n^2-n)=\Theta(n^2).$$

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