Part II

Foundations

3 Goals

- Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.
- Learn how to analyze and judge the efficiency of algorithms.
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What do you measure?

▶ Memory requirement

▶ Running time

▶ Number of comparisons

▶ Number of multiplications

▶ Number of hard-disc accesses

▶ Program size

▶ Power consumption

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How do you measure?

- ▶ Implementing and testing on representative inputs
    - ▸ How do you choose your inputs?
    - ▸ May be very time-consuming.
    - ▸ Very reliable results if done correctly.
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- ▶ Theoretical analysis in a specific model of computation.

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- Theoretical analysis in a specific model of computation.
  - Gives asymptotic bounds like $O(n^2)$.
  - Very robust also for experimental data.
  - Often more convenient: e.g. function instead of large value table.

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The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$
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Suppose n numbers from the interval $\{1, \ldots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), . . .

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Turing Machine

▶ Very simple model of computation.

▶ Only the "current" memory location can be altered.

▶ Very good model for discussing computabiliy, or polynomial vs. exponential time.

▶ Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.

⟹ Not a good model for developing efficient algorithms.

control unit

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state holds program and can act as constant size memory

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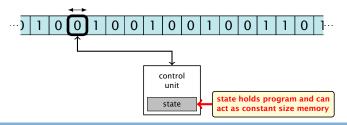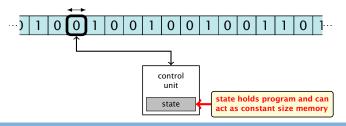
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Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).

- Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \dots$.

- Registers hold integers.

- Indirect addressing.

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  - $R[j] := 4$
- indirect addressing
  - $R[j] := R[R[i]]$
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- ▶ branching (including loops) based on comparisons

  - ▶ jump x
    jumps to position x in the program;
    sets instruction counter to x;
    reads the next operation to perform from register $R[x]$
  - ▶ jumpz $x\ R[i]$
    jump to x if $R[i] = 0$
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- **uniform** cost model
  Every operation takes time 1.
- logarithmic cost model
  The cost depends on the content of memory cells:

Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed w, where usually $w = \log_2 n$.

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Example 2

Algorithm 1 RepeatedSquaring(n)
1: $r \leftarrow 2;$
2: **for** $i = 1 \rightarrow n$ **do**
3: $\qquad r \leftarrow r^2$
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There are different types of complexity bounds:

▶ best-case complexity:

$$C_{\text{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

▶ worst-case complexity:

$$C_{\text{wc}}(n) := \max\{C(x) \mid |x| = n\}$$

Usually moderately easy to analyze; sometimes too pessimistic.

▶ average case complexity:

$$C_{\text{avg}}(n) := \frac{1}{|I_n|} \sum_{|x|=n} C(x)$$

more general: probability measure μ

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There are different types of complexity bounds:

▶ amortized complexity:
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▶ randomized complexity:
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Formal Definition

Let f denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

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Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0 \; \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
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The expressions also hold for Ω. Note that this means that
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Comments

▶ Do not use asymptotic notation within induction proofs.

▶ For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.

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6 Recurrences

Algorithm 2 mergesort(list L)

1: $s \leftarrow \text{size}(L)$
2: **if** $s \leq 1$ **return** L
3: $L_1 \leftarrow L[1 \cdots \lfloor \frac{s}{2} \rfloor]$
4: $L_2 \leftarrow L[\lceil \frac{s}{2} \rceil \cdots n]$
5: mergesort(L_1)
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7: $L \leftarrow \text{merge}(L_1, L_2)$
8: **return** L

This algorithm requires

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

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Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Assume that instead we had

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Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

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  We prove it for n:

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  Suppose statem. is true for $n' \in \{2, \ldots, n - 1\}$, and $n \geq 16$.
  We prove it for n:

  $$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.
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  $$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \end{aligned}$$

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  Suppose statem. is true for $n' \in \{2, \ldots, n-1\}$, and $n \geq 16$. We prove it for n:

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  Hence, statement is true if we choose $d \geq c$.

6.1 Guessing+Induction

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If we do not do this we instead consider the following recurrence:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

6.1 Guessing+Induction

We also make a guess of $T(n) \le dn \log n$ and get

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$$\leq dn \log n - 0.33dn + cn$$

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We also make a guess of $T(n) \leq dn \log n$ and get

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$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of d.

6.2 Master Theorem

Lemma 4

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.
If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.
If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Case 3.
If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
$af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The running time of a recursive algorithm can be visualized by a recursion tree:

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6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \ .$$

Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad &= c n^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i
\end{aligned}
$$

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$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}}$$

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$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

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$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

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Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1}n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right)n^{\log_b(a)}$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1}-1}{q-1}} = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1}n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \le \left(\frac{c}{b^\epsilon - 1} + 1\right)n^{\log_b(a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$T(n) - n^{\log_b a}$$

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

Case 2. Now suppose that $f(n) \leq c n^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= c n^{\log_b a} \log_b n
\end{aligned}
$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,
$$
T(n) = \mathcal{O}(n^{\log_b a} \log_b n)
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= c n^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k
\end{aligned}
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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k
\end{aligned}
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$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$ $= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \frac{1}{k}\ell^{k+1}$$

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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
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&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
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$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad = c n^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

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$$= c n^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \quad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&= \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
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$$q < 1 : \sum_{i=0}^{n} q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}} \quad \leq \frac{1}{1 - c} f(n) + \mathcal{O}(n^{\log_b a})$$

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\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&= \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1 - q^{n+1}}{1-q} \leq \frac{1}{1-q}}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$ $\leq \dfrac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$

Hence,

$$T(n) \leq \mathcal{O}(f(n)) \qquad \boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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For this we first need to be able to add two integers A and B:

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$$\begin{array}{ccccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array} \quad A$$

$$\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array} \quad B$$

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\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\
\end{array}
\qquad
\begin{array}{c}
A \\[1ex]
B
\end{array}
$$

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$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
 & & & & & & & & 0
\end{array}
\quad
\begin{array}{l}
A \\
B \\
\end{array}
$$

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$$
\begin{array}{cccccccccc}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
\hline
 & & & & & & & & {\scriptstyle 1} & & & \\
 & & & & & & & & 0 & & &
\end{array}
$$

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\hline
 & & & & & & & 0 & 0 &
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\textbf{1} & \textbf{1} & \textbf{0} & \textbf{1} & \textbf{1} & \textbf{0} & \textbf{1} & \textbf{0} & \textbf{1} \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
 & & & & & & \textbf{0} & \textbf{0} & \textbf{0}
\end{array} \quad \begin{array}{c} A \\ B \end{array}
$$

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$$
\begin{array}{ccccccccc}
\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & A \\
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & B \\
\hline
 & & & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & &
\end{array}
$$

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Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.
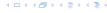
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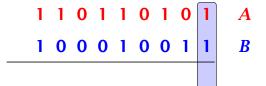For this we first need to be able to add two integers A and B:

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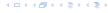For this we first need to be able to add two integers A and B:

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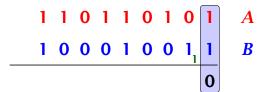For this we first need to be able to add two integers A and B:

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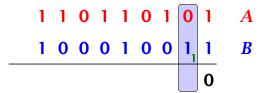
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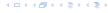For this we first need to be able to add two integers A and B:

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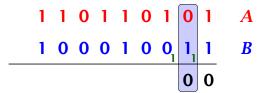
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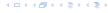
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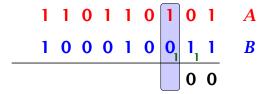
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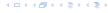
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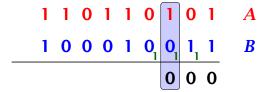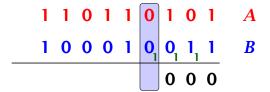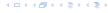For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
\end{array}
\quad A
$$

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
{}_{1} & {}_{0} & {}_{0} & {}_{1} & {}_{1} & {}_{0} & {}_{1} & {}_{1} & {}_{1} \\
\end{array}
\quad B
$$

$$
\begin{array}{ccccccccc}
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$\begin{array}{c} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \end{array}$$

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$1\ 0\ 0\ 0\ 1 \times 1\ 0\ \boxed{1}\ 1$$
$$1\ 0\ 0\ 0\ 1$$
$$0$$

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{ccccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & 1 & 0 & 0 & 0 & 1 & & \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & & \\
\end{array}
$$

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1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
$$

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$$1\ 0\ 0\ 0\ 1 \times 1\ \boxed{0}\ 1\ 1$$

$$1\ 0\ 0\ 0\ 1$$
$$1\ 0\ 0\ 0\ 1\ 0$$
$$0\ 0$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & 1 & 0 & 0 & 0 & 1 & & \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & & \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & & \\
\end{array}
$$

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\begin{array}{ccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & 1 & 0 & 0 & 0 & 1 \\
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 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 \\
\end{array}
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$$
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
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$$

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\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccc}
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \times & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\
\hline
 & & & & & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\
 & & & & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
 & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 & & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

```
1 0 0 0 1 × 1 0 1 1
─────────────────────
          1 0 0 0 1
        1 0 0 0 1 0
      0 0 0 0 0 0 0
    1 0 0 0 1 0 0 0
─────────────────────
  1 0 1 1 1 0 1 1
```

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.

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\hline
& & & & & 1 & 0 & 0 & 0 & 1 \\
& & & & 1 & 0 & 0 & 0 & 1 & 0 \\
& & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
& & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
$$

Time requirement:

▶ Computing intermediate results: $\mathcal{O}(nm)$.

▶ Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

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| b_n | $\cdots$ | b_0 | $\times$ | a_n | $\cdots$ | a_0 |

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{b_n \quad \cdots \quad b_{\frac{n}{2}} \ b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \ \times \ \boxed{a_n \quad \cdots \quad a_{\frac{n}{2}} \ a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

Example: Multiplying Two Integers

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Suppose that integers A and B are of length $n = 2^k$, for some k.

B_1	B_0	$\times$	A_1	A_0

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Suppose that integers A and B are of length $n = 2^k$, for some k.

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 \cdot B_0$$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**
2:        **return** $a_0 \cdot b_0$
3: split A into A_0 and A_1
4: split B into B_0 and B_1
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
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Example: Multiplying Two Integers

Algorithm 3 mult(A, B)	
1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
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5: $Z_2 \leftarrow$ mult(A_1, B_1)	
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Example: Multiplying Two Integers

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5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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⇒ Not better then the "school method".

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We can use the following identity to compute Z_1:

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$$= (A_0 + A_1) \cdot (B_0 + B_1) - A_1 B_1 - A_0 B_0$$

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$
\begin{aligned}
Z_1 &= A_1 B_0 + A_0 B_1 \\
&= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0}
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Hence,

Algorithm 4 mult(A, B)

1: **if** $|A| = |B| = 1$ **then**
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3: split A into A_0 and A_1
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We can use the following identity to compute Z_1:

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1: **if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
2:       **return** $a_0 \cdot b_0$ $\mathcal{O}(1)$
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Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

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Z_1 &= A_1 B_0 + A_0 B_1 \\
&= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0}
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Hence,

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1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
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We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$

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Again we are in Case 1. We get a running time of
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A huge improvement over the "school method".

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6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

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- $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.

- The recurrence is linear as there are no products of $T[n]$'s.

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Observations:

- The solution $T[0], T[1], T[2], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[0], \ldots, T[k-1]$.

- In fact, any k consecutive values completely determine the solution.

- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.

- Then pick the right one by analyzing boundary conditions.

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The Homogenous Case

The solution space

$$S = \Big\{ T = T[0], T[1], T[2], \ldots \ \Big| \ T \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $T_1, T_2 \in S$, then also $\alpha T_1 + \beta T_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?
We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

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Lemma 5

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n \ .$$

Proof.
There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

The Homogenous Case

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We show that the column vectors are linearly independent. Then the above equation has a solution.

The Homogenous Case

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The Homogenous Case

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  assume for contradiction that there exist α_i's with

$$\alpha_1 \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix} + \cdots + \alpha_k \begin{pmatrix} \lambda_k \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix} = 0$$

and not all $\alpha_i = 0$. $\qquad \square$

The Homogenous Case

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This we show by induction:

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and not all $\alpha_i = 0$. **Then all $\alpha_i \neq 0$!** $\qquad\square$

The Homogeneous Case

$$\alpha_1 \begin{pmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix} + \cdots + \alpha_k \begin{pmatrix} \lambda_1 \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix} = 0$$

The Homogeneous Case

$$\alpha_1 \underbrace{\begin{pmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix}}_{v_1 :=} + \cdots + \alpha_k \underbrace{\begin{pmatrix} \lambda_1 \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix}}_{v_k :=} = 0$$

The Homogeneous Case

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$v_1 :=$ (top entry λ_1), $\quad \lambda_1 v_1 =$ (bottom entry λ_1^k)

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The Homogeneous Case

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v_1 := \begin{pmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix} \qquad v_k := \begin{pmatrix} \lambda_1 \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix}
$$

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$\lambda_1 v_1 = \lambda_1^k$ $\qquad$ $\lambda_k v_k = \lambda_k^k$

This means that

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\sum_{i=1}^{k} \alpha_i v_i = 0 \text{ and } \sum_{i=1}^{k} \lambda_i \alpha_i v_i = 0
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The Homogeneous Case

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Hence,

$$\sum_{i=1}^{k-1} \alpha_i v_i + \alpha_k v_k = 0 \text{ and } -\frac{1}{\lambda_k} \sum_{i=1}^{k-1} \lambda_i \alpha_i v_i = \alpha_k v_k$$

The Homogeneous Case

This gives that

$$\sum_{i=1}^{k-1} (1 - \frac{\lambda_i}{\lambda_k}) \alpha_i v_i = 0 \ .$$

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This is a contradiction as the v_i's are linearly independent because of induction hypothesis.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P(\lambda)\lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q(\lambda)(\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

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Suppose λ_i has multiplicity j. We know that

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(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \cdots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue $j - 1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j - 1$.

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The Homogeneous Case

Lemma 6

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0 T[n] + c_1 T[n-1] + \cdots + c_k T[n-k] = 0$$

Let λ_i, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

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The Inhomogeneous Case

Example:
$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,
$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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$T[1] = 2$ gives $1 + \beta = 2 \implies \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2$$

Difference:

$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

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The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

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$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$

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Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$
$$= 2T[n-2] - T[n-3] + 2n - 3$$

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and so on...

6.4 Generating Functions

Definition 7 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n=0}^{\infty} a_n z^n;$$

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1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

$$F(z) = 1 \, .$$

2. The generating function of the sequence $(1, 1, 1, \ldots)$ is

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6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$.

▶ **Equality:** f and g are equal if $a_n = b_n$ for all n.

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▶ **Multiplication:** $f \cdot g := \sum_{n=0}^{\infty} c_n z^n$ with $c = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

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The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

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What does $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n=0}^{\infty} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n=0}^{\infty} z^n \right) = 1 \ .$$

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^2}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

6.4 Generating Functions

We know

$$\sum_{n \geq 0} y^n = \frac{1}{1 - y}$$

Hence,

$$\sum_{n \geq 0} a^n z^n = \frac{1}{1 - az}$$

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Suppose we have again the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

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&= zA(z) + \sum_{n \geq 0} z^n \\
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Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	generating function
1	$\dfrac{1}{1-z}$
$n+1$	$\dfrac{1}{(1-z)^2}$
$\binom{n+k}{n}$	$\dfrac{1}{(1-z)^{k+1}}$
n	$\dfrac{z}{(1-z)^2}$
a^n	$\dfrac{1}{1-az}$
n^2	$\dfrac{z(1+z)}{(1-z)^3}$
$\frac{1}{n!}$	$\dfrac{z(1+z)}{(1-z)^3}$

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f_{n-k} $(n \geq k);$ 0 otw.	$z^k F$
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6. The coefficients of the resulting power series are the a_n.

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4} 3^n - \frac{1}{2} n - \frac{3}{4}$.

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Example 9

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,.$$

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