6 Recurrences

Algorithm 2 mergesort(list L)

$s \leftarrow \mathrm{size}(L)$
if $s \leq 1$ **return** L
$L_1 \leftarrow L[1 \cdots \lfloor \frac{s}{2} \rfloor]$
$L_2 \leftarrow L[\lceil \frac{s}{2} \rceil \cdots n]$
$\mathrm{mergesort}(L_1)$
$\mathrm{mergesort}(L_2)$
$L \leftarrow \mathrm{merge}(L_1, L_2)$
return L

This algorithm requires

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

For this we need to solve the recurrence.

Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Assume that instead we had

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

6.1 Guessing+Induction

Suppose we guess $T(n) \le dn \log n$ for a constant d. Then

$$\begin{aligned} T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\ &\le 2\left(\frac{n}{2}\log\frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn\log n + (c-d)n \\ &= dn\log n \end{aligned}$$

if we choose $d \ge c$.

Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

- **base case** $(2 \leq n < 16)$: true if we choose $d \geq b$.
- **induction step** $2 \ldots n-1 \to n$:

  Suppose statem. is true for $n' \in \{2, \ldots, n-1\}$, and $n \geq 16$.
  We prove it for n:

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn \log n + (c-d)n \\ &= dn \log n \end{aligned}$$

- Note that this proves the statement for $n \in \mathbb{N}_{\geq 2}$, as the statement is wrong for $n = 1$.
- The base case is usually omitted, as it is the same for different recurrences.

Hence, statement is true if we choose $d \geq c$.

6.1 Guessing+Induction

Why did we change the recurrence by getting rid of the ceiling?

If we do not do this we instead consider the following recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

6.1 Guessing+Induction

We also make a guess of $T(n) \le dn \log n$ and get

$$
\begin{aligned}
T(n) &\le 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\
&\le 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\
\boxed{\left\lceil \tfrac{n}{2} \right\rceil \le \tfrac{n}{2} + 1} \quad &\le 2(d(n/2+1)\log(n/2+1)) + cn \\
\boxed{\tfrac{n}{2} + 1 \le \tfrac{9}{16}n} \quad &\le dn \log\left(\frac{9}{16}n\right) + 2d\log n + cn \\
\boxed{\log \tfrac{9}{16}n = \log n + (\log 9 - 4)} \quad &= dn\log n + (\log 9 - 4)dn + 2d\log n + cn \\
\boxed{\log n \le \tfrac{n}{4}} \quad &= dn\log n + (\log 9 - 3.5)dn + cn \\
&\le dn\log n - 0.33dn + cn \\
&\le dn\log n
\end{aligned}
$$

for a suitable choice of d.

6.2 Master Theorem

Lemma 4

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

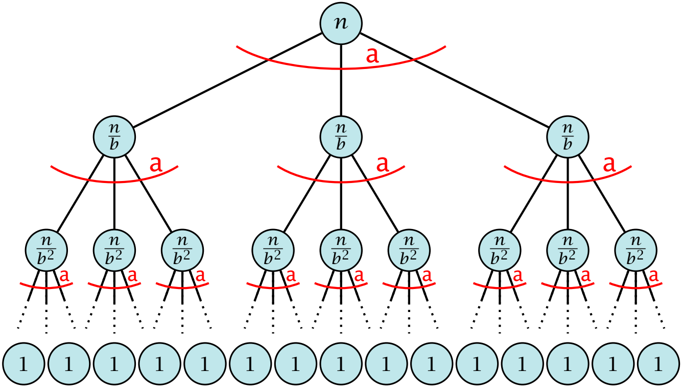
Note that the cases do not cover all possibilities.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



$$f(n)$$

$$af(\tfrac{n}{b})$$

$$a^2 f(\tfrac{n}{b^2})$$

$$a^{\log_b n}$$

$$=$$

$$n^{\log_b a}$$

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \ .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a-\epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}$$

$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad = cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(b^{\epsilon}\right)^i$$

$$\boxed{\textstyle\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad = cn^{\log_b a-\epsilon}(b^{\epsilon\log_b n} - 1)/(b^{\epsilon} - 1)$$

$$= cn^{\log_b a-\epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1)$$

$$= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})$$

Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b(a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$
T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}
$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$
T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}
$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \frac{1}{k}\ell^{k+1} \\
&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\end{aligned}
$$

$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&= \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$\Rightarrow T(n) = \Theta(f(n))$.

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{cccccccccccc}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & & & \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

- This is also nown as the "school method" for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \le 2n$ bits.

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

B_1	B_0	$\times$	A_1	A_0

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 \cdot B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $\|A\| = \|B\| = 1$ **then**	$\mathcal{O}(1)$
return $a_0 \cdot b_0$	$\mathcal{O}(1)$
split A into A_0 and A_1	$\mathcal{O}(n)$
split B into B_0 and B_1	$\mathcal{O}(n)$
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$	$T(\frac{n}{2})$
$Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$	$T(\frac{n}{2})$
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\quad T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

$\Rightarrow$ Not better then the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1)\cdot(B_0 + B_1) - \underbrace{A_1B_1}_{= Z_2} - \underbrace{A_0B_0}_{= Z_0} \end{aligned}$$

Hence,

Algorithm 4 mult(A, B)	
if $\lvert A\rvert = \lvert B\rvert = 1$ **then**	$\mathcal{O}(1)$
return $a_0 \cdot b_0$	$\mathcal{O}(1)$
split A into A_0 and A_1	$\mathcal{O}(n)$
split B into B_0 and B_1	$\mathcal{O}(n)$
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$	$T(\frac{n}{2})$
$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
$Z_1 \leftarrow \mathrm{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2})$
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \enspace .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a-\epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a}\log^k n)$ $\quad T(n) = \Theta(n^{\log_b a}\log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a+\epsilon})$ $\quad T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

- $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.
- The recurrence is linear as there are no products of $T[n]$'s.
- If $f(n) = 0$ then the recurrence relation becomes a linear, homogenous recurrence relation of order k.

6.3 The Characteristic Polynomial

Observations:

- The solution $T[0], T[1], T[2], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[0], \ldots, T[k-1]$.
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \Big\{ T = T[0], T[1], T[2], \ldots \;\Big|\; T \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $T_1, T_2 \in S$, then also $\alpha T_1 + \beta T_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i.

The Homogenous Case

Lemma 5

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \dots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n \ .$$

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$
\begin{array}{ccccccccc}
\alpha_1 \cdot \lambda_1 & + & \alpha_2 \cdot \lambda_2 & + & \cdots & + & \alpha_k \cdot \lambda_k & = & T[1] \\
\alpha_1 \cdot \lambda_1^2 & + & \alpha_2 \cdot \lambda_2^2 & + & \cdots & + & \alpha_k \cdot \lambda_k^2 & = & T[2] \\
 & & & & \vdots & & & & \\
\alpha_1 \cdot \lambda_1^k & + & \alpha_2 \cdot \lambda_2^k & + & \cdots & + & \alpha_k \cdot \lambda_k^k & = & T[k]
\end{array}
$$

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.

The Homogenous Case

Proof (cont.).

This we show by induction:

- **base case** ($k = 1$):
  A vector (λ_i), $\lambda_i \neq 0$ is linearly independent.
- **induction step** ($k \to k + 1$):
  assume for contradiction that there exist α_i's with

$$\alpha_1 \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix} + \cdots + \alpha_k \begin{pmatrix} \lambda_k \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix} = 0$$

and not all $\alpha_i = 0$. **Then all $\boldsymbol{\alpha_i \neq 0}$!**

□

The Homogeneous Case

$$
\begin{array}{r}
v_1 := \\ \\ \\ \\ \lambda_1 v_1 =
\end{array}
\alpha_1 \begin{pmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix}
+ \cdots +
\begin{array}{r}
v_k := \\ \\ \\ \\ \lambda_k v_k =
\end{array}
\alpha_k \begin{pmatrix} \lambda_1 \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix} = 0
$$

This means that

$$\sum_{i=1}^{k} \alpha_i v_i = 0 \text{ and } \sum_{i=1}^{k} \lambda_i \alpha_i v_i = 0$$

Hence,

$$\sum_{i=1}^{k-1} \alpha_i v_i + \alpha_k v_k = 0 \text{ and } -\frac{1}{\lambda_k} \sum_{i=1}^{k-1} \lambda_i \alpha_i v_i = \alpha_k v_k$$

The Homogeneous Case

This gives that

$$\sum_{i=1}^{k-1}(1-\frac{\lambda_i}{\lambda_k})\alpha_i v_i = 0 \enspace .$$

This is a contradiction as the v_i's are linearly independent because of induction hypothesis.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P(\lambda)\lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q(\lambda)(\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$c_0 \underbrace{n\lambda_i^n}_{T[n]} + c_1 \underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k \underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

The Homogeneous Case

Suppose λ_i has multiplicity j. We know that

$$c_0 n\lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

The Homogeneous Case

Lemma 6

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0 T[n] + c_1 T[n-1] + \cdots + c_k T[n-k] = 0$$

Let λ_i, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$
\begin{aligned}
T[0] &= 0 \\
T[1] &= 1 \\
T[n] &= T[n-1] + T[n-2] \text{ for } n \geq 2
\end{aligned}
$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)^n+\beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$T[0]=0$ gives $\alpha+\beta=0$.

$T[1]=1$ gives

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1 \Longrightarrow \alpha-\beta=\frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \quad ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r+1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

Difference:

$$\begin{aligned} T[n] - T[n-1] =& 2T[n-1] - T[n-2] + 2n - 1 \\ & - 2T[n-2] + T[n-3] - 2n + 3 \end{aligned}$$

$$T[n] = 3T[n-1] - 3T[n-2] + T[n-3] + 2$$

and so on...

6.4 Generating Functions

Definition 7 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n=0}^{\infty} a_n z^n;$$

- exponential generating function (exponentielle Erzeugendenfunktion) is

$$F(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n.$$

6.4 Generating Functions

Example 8

1. The generating function of the sequence $(1, 0, 0, \ldots)$ is
$$F(z) = 1 .$$

2. The generating function of the sequence $(1, 1, 1, \ldots)$ is
$$F(z) = \frac{1}{1-z} .$$

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n=0}^{\infty} (a_n + b_n) z^n$.
- **Multiplication:** $f \cdot g := \sum_{n=0}^{\infty} c_n z^n$ with $c = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

6.4 Generating Functions

What does $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n=0}^{\infty} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n=0}^{\infty} z^n\right) = 1 \ .$$

This is well-defined.

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \ .$$

We can compute the derivative:

$$\underbrace{\sum_{n\ge 1} nz^{n-1}}_{\sum_{n=0}^{\infty}(n+1)z^n} = \frac{1}{(1-z)^2}$$

Hence, the generating function of the sequence $a_n = n+1$ is $1/(1-z)^2$.

6.4 Generating Functions

We can repeat this

$$\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2} \enspace .$$

Derivative:

$$\underbrace{\sum_{n\geq 1} n(n+1)z^{n-1}}_{\sum_{n=0}^{\infty}(n+1)(n+2)z^n} = \frac{2}{(1-z)^3}$$

Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^2}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n\geq k} n(n-1)\dots(n-k+1)z^{n-k} = \sum_{n\geq 0}(n+k)\dots(n+1)z^n$$
$$= \frac{k!}{(1-z)^{k+1}} \ .$$

Hence:

$$\sum_{n\geq 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}} \ .$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

6.4 Generating Functions

$$\begin{aligned}\sum_{n\geq 0} nz^n &= \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z} \\ &= \frac{z}{(1-z)^2}\end{aligned}$$

The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

6.4 Generating Functions

Suppose we have again the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$\begin{aligned}
A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \\
&= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n \\
&= zA(z) + \sum_{n\geq 0} z^n \\
&= zA(z) + \frac{1}{1-z}
\end{aligned}$$

6.4 Generating Functions

Solving for $A(z)$ gives

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n\geq 0} (n+1) z^n$$

Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	generating function
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{n}$	$\frac{1}{(1-z)^{k+1}}$
n	$\frac{z}{(1-z)^2}$
a^n	$\frac{1}{1-az}$
n^2	$\frac{z(1+z)}{(1-z)^3}$
$\frac{1}{n!}$	$\frac{z(1+z)}{(1-z)^3}$

Some Generating Functions

n-th sequence element	generating function
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \quad (n \geq k); \quad 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1-z}$
nf_n	$z\dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$
$c^n f_n$	$F(cz)$

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
5. Write $f(z)$ as a formal power series.
   Techniques:
   - partial fraction decomposition (Partialbruchzerlegung)
   - lookup in tables
6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}, a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n\geq 1} a_n z^n$$

2. Plug in:

$$A(z) = 1 + \sum_{n\geq 1} (2a_{n-1}) z^n$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$\begin{aligned}
A(z) &= 1 + \sum_{n\geq 1}(2a_{n-1})z^n \\
&= 1 + 2z\sum_{n\geq 1}a_{n-1}z^{n-1} \\
&= 1 + 2z\sum_{n\geq 0}a_n z^n \\
&= 1 + 2z \cdot A(z)
\end{aligned}$$

4. Solve for $A(z)$.

$$A(z) = \frac{1}{1-2z}$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

5. Rewrite $f(n)$ as a power series:

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned}
A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} a_n z^n \\
&= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n \\
&= 1 + 3z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} n z^n \\
&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
\end{aligned}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n,\ a_0 = 1$

5. Write $f(z)$ as a formal power series:

   We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2} \stackrel{!}{=} \frac{A}{1-3z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

   This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

   which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$
\begin{aligned}
A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\
&= \frac{7}{4} \cdot \sum_{n\geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\geq 0} z^n - \frac{1}{2} \cdot \sum_{n\geq 0} (n+1) z^n \\
&= \sum_{n\geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \Big) z^n
\end{aligned}
$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 9

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 2 \\ f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,. \end{aligned}$$

Define

$$g_n := \log f_n \,.$$

Then

$$\begin{aligned} g_n &= g_{n-1} + g_{n-2} \text{ for } n \geq 2 \\ g_1 &= \log 2 = 1,\; g_0 = 0 \text{ (fÃČÅŠr } \log = \log_2 \text{)} \\ g_n &= F_n \text{ (n-th Fibonacci number)} \\ f_n &= 2^{F_n} \end{aligned}$$

6.5 Transformation of the Recurrence

Example 10

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n;\ \text{for } n = 2^k\ ;$$

Define

$$g_k := f_{2^k}\ .$$

6.5 Transformation of the Recurrence

Example 10

Then:

$$\begin{aligned} g_0 &= 1 \\ g_k &= 3g_{k-1} + 2^k,\ k \geq 1 \end{aligned}$$

We get,

$$\begin{aligned} g_k &= 3^{k+1} - 2^{k+1},\ \text{hence} \\ f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log 3})^k - 2 \cdot 2^k \\ &= 3(2^k)^{\log 3} - 2 \cdot 2^k \\ &= 3n^{\log 3} - 2n\,. \end{aligned}$$