How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

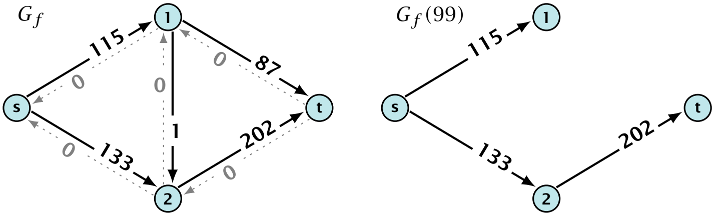
Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.

Capacity Scaling

Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter Δ.
- $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.



Capacity Scaling

Algorithm 46 maxflow(G, s, t, c)

```
foreach e ∈ E do f_e ← 0;
Δ ← 2^⌈log₂ C⌉
while Δ ≥ 1 do
    G_f(Δ) ← Δ-residual graph
    while there is augmenting path P in G_f(Δ) do
        f ← augment(f, c, P)
        update(G_f(Δ))
    Δ ← Δ/2
return f
```

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

- because of integrality we have $G_f(1) = G_f$
- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.

Capacity Scaling

Lemma 60

There are $\lceil \log C \rceil$ iterations over Δ.

Proof: obvious.

Lemma 61

Let f be the flow at the end of a Δ-phase. Then the maximum flow is smaller than $\mathrm{val}(f) + 2m\Delta$.

Proof: less obvious, but simple:

- An s-t cut in $G_f(\Delta)$ gives me an upper bound on the amount of flow that my algorithm can still add to f.
- The edges that currently have capacity at most Δ in G_f form an s-t cut with capacity at most $2m\Delta$.

Capacity Scaling

Lemma 62

There are at most $2m$ augmentations per scaling-phase.

Proof:

- Let f be the flow at the end of the previous phase.
- $\mathrm{val}(f^*) \leq \mathrm{val}(f) + 2m\Delta$
- each augmentation increases flow by Δ.

Theorem 63

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.