6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

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Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

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  Hence, statement is true if we choose $d \ge c$.

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If we do not do this we instead consider the following recurrence:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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for a suitable choice of d.