10 Karmarkars Algorithm

We want to solve the following linear program:

- $\min v = c^t x$ subject to $Ax = 0$ and $x \in \Delta$.
- Here $\Delta = \{x \in \mathbb{R}^n \mid e^t x = 1, x \geq 0\}$ with $e^t = (1, \ldots, 1)$ denotes the standard simplex in $\mathbb{R}^n$.

Further assumptions:

1. A is an $m \times n$-matrix with rank m.
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- Add $-(\sum_i x_i)b_i = -b_i$ to every constraint. ⇒ vector b is 0
- If A does not have full column rank we can delete constraints (or conclude that the LP is infeasible). ⇒ A has full row rank

We still need to make e/n feasible.

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The algorithm computes (strictly) feasible interior points $\bar{x}^{(0)} = \frac{e}{n}, x^{(1)}, x^{(2)}, \ldots$ with

$$c^t x^k \leq 2^{-\Theta(L)} c^t x^0$$

For $k = \Theta(L)$. A point x is strictly feasible if $x > 0$.

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Iteration:

1. Distort the problem by mapping the simplex onto itself so that the current point $\bar{x}$ moves to the center.
2. Project the optimization direction c onto the feasible region. Determine a distance to travel along this direction such that you do not leave the simplex (and you do not touch the border). $\hat{x}$ is the point you reached.
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The Transformation

Let $\bar{Y} = \mathrm{diag}(\bar{x})$ the diagonal matrix with entries $\bar{x}$ on the diagonal.

Define

$$F_{\bar{x}} : x \mapsto \frac{\bar{Y}^{-1}x}{e^t\bar{Y}^{-1}x} \enspace .$$

The inverse function is

$$F_{\bar{x}}^{-1} : \hat{x} \mapsto \frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}} \enspace .$$

Note that $\bar{x} > 0$ in every coordinate. Therefore the above is well defined.

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Note that $\bar{x} > 0$ in every coordinate. Therefore the above is well defined.

Properties

$F_{\bar{x}}^{-1}$ really is the inverse of $F_{\bar{x}}$:

$$F_{\bar{x}}(F_{\bar{x}}^{-1}(\hat{x})) = \frac{\bar{Y}^{-1}\frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}}}{e^t\bar{Y}^{-1}\frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}}} = \frac{\hat{x}}{e^t\hat{x}} = \hat{x}$$

because $\hat{x} \in \Delta$.

Note that in particular every $\hat{x} \in \Delta$ has a preimage (Urbild) under $F_{\bar{x}}$.

Properties

$\bar{x}$ is mapped to e/n

$$F_{\bar{x}}(\bar{x}) = \frac{\bar{Y}^{-1}\bar{x}}{e^t\bar{Y}^{-1}\bar{x}} = \frac{e}{e^t e} = \frac{e}{n}$$

Properties

A unit vectors e_i is mapped to itself:

$$F_{\bar{x}}(e_i) = \frac{\bar{Y}^{-1}e_i}{e^t\bar{Y}^{-1}e_i} = \frac{(0,\dots,0,\bar{x}_i,0,\dots,0)^t}{e^t(0,\dots,0,\bar{x}_i,0,\dots,0)^t} = e_i$$

Properties

All nodes of the simplex are mapped to the simplex:

$$F_{\bar{x}}(x) = \frac{\bar{Y}^{-1}x}{e^t\bar{Y}^{-1}x} = \frac{\left(\frac{x_1}{\bar{x}_1},\dots,\frac{x_n}{\bar{x}_n}\right)^t}{e^t\left(\frac{x_1}{\bar{x}_1},\dots,\frac{x_n}{\bar{x}_n}\right)^t} = \frac{\left(\frac{x_1}{\bar{x}_1},\dots,\frac{x_n}{\bar{x}_n}\right)^t}{\sum_i \frac{x_i}{\bar{x}_i}} \in \Delta$$

The Transformation

Easy to check:

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After the transformation we have the problem

$$\min\{c^t F_{\bar{x}}^{-1}(x) \mid AF_{\bar{x}}^{-1}(x) = 0; x \in \Delta\}$$

This holds since the back-transformation “reaches” every point in Δ (i.e. $F_{\bar{x}}^{-1}(\Delta) = \Delta$).

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This holds since the back-transformation “reaches” every point in Δ (i.e. $F_{\bar{x}}^{-1}(\Delta) = \Delta$).

Since the optimum solution is 0 this problem is the same as

$$\min\{\hat{c}^t x \mid \hat{A}x = 0, x \in \Delta\}$$

with $\hat{c} = \bar{Y}^t c = \bar{Y} c$ and $\hat{A} = A\bar{Y}$.

We still need to make e/n feasible.

- We know that our LP is feasible. Let $\bar{x}$ be a feasible point.
- Apply $F_{\bar{x}}$, and solve

$$\min\{\hat{c}^t x \mid \hat{A}x = 0; x \in \Delta\}$$

- The feasible point is moved to the center.

10 Karmarkars Algorithm

When computing $\hat{x}$ we do not want to leave the simplex or touch its boundary (why?).

For this we compute the radius of a ball that completely lies in the simplex.

$$B\left(\frac{e}{n},\rho\right) = \left\{x \in \mathbb{R}^n \mid \left\|x - \frac{e}{n}\right\| \leq \rho\right\} \quad .$$

We are looking for the largest radius r such that

$$B\left(\frac{e}{n},r\right) \cap \left\{x \mid e^t x = 1\right\} \subseteq \Delta.$$

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This holds for $r = \|\frac{e}{n} - (e - e_1)\frac{1}{n-1}\|$. ($r$ is the distance between the center e/n and the center of the $(n-1)$-dimensional simplex obtained by intersecting a side ($x_i = 0$) of the unit cube with Δ.)

This gives $r = \frac{1}{\sqrt{n(n-1)}}$.

Now we consider the problem

$$\min\{\hat{c}^t x \mid \hat{A}x = 0, x \in B(e/n, r) \cap \Delta\}$$

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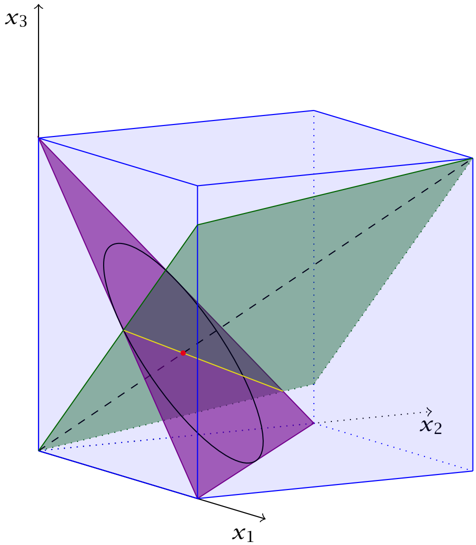
The Simplex



10 Karmarkars Algorithm

Ideally we would like to go in direction of $-\hat{c}$ (starting from the center of the simplex).

However, doing this may violate constraints $\hat{A}x = 0$ or the constraint $x \in \Delta$.

Therefore we first project $\hat{c}$ on the nullspace of

$$B = \begin{pmatrix} \hat{A} \\ e^t \end{pmatrix}$$

We use

$$P = I - B^t(BB^t)^{-1}B$$

Then

$$\hat{d} = P\hat{c}$$

is the required projection.

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We get the new point

$$\hat{x}(\rho) = \frac{e}{n} - \rho \frac{\hat{d}}{\|d\|}$$

for $\rho < r$.

Choose $\rho = \alpha r$ with $\alpha = 1/4$.

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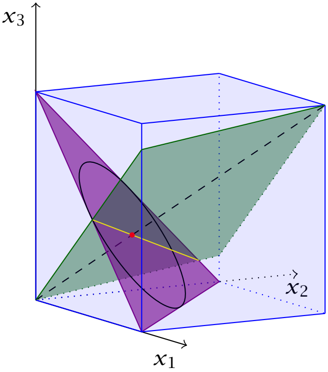
10 Karmarkars Algorithm

Iteration of Karmarkars algorithm:

- Current solution $\bar{x}$. $\bar{Y} := \text{diag}(\bar{x}_1, \dots, \bar{x}_n)$.
- Transform the problem via $F_{\bar{x}}(x) = \frac{\bar{Y}^{-1}x}{e^t \bar{Y}^{-1}x}$. Let $\hat{c} = \bar{Y}c$, and $\hat{A} = A\bar{Y}$.
- Compute
$$d = (I - B^t(BB^t)^{-1}B)\hat{c} \;\; ,$$
where $B = \begin{pmatrix} \hat{A} \\ e^t \end{pmatrix}$.
- Set
$$\hat{x} = \frac{e}{n} - \rho \frac{d}{\|d\|} \;\; ,$$
with $\rho = \alpha r$ with $\alpha = 1/4$ and $r = 1/\sqrt{n(n-1)}$.
- Compute $\bar{x}_{\text{new}} = F_{\bar{x}}^{-1}(\hat{x})$.

The Simplex



Lemma 2

The new point $\hat{x}$ in the transformed space is the point that minimizes the cost $\hat{c}^t x$ among all feasible points in $B(\frac{e}{n}, \rho)$.

Proof: Let z be another feasible point in $B(\frac{e}{n}, \rho)$.

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Further,

$$(\hat{c} - d)^t$$

Proof: Let z be another feasible point in $B(\frac{e}{n},\rho)$.

As $\hat{A}z = 0$, $\hat{A}\hat{x} = 0$, $e^t z = 1$, $e^t\hat{x} = 1$ we have

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which means that the cost-difference between $\hat{x}$ and z is the same measured w.r.t. the cost-vector $\hat{c}$ or the projected cost-vector d.

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as $\frac{e}{n} - z$ is a vector of length at most ρ.

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This gives $d(\hat{x}-z) \le 0$ and therefore $\hat{c}\hat{x} \le \hat{c}z$.

In order to measure the progress of the algorithm we introduce a potential function f:

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- The function f is invariant to scaling (i.e., $f(kx) = f(x)$).
- The potential function essentially measures cost (note the term $n \ln(c^t x)$) but it penalizes us for choosing x_j values very small (by the term $-\sum_j \ln(x_j)$; note that $-\ln(x_j)$ is always positive).

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This means the potential of a point in the transformed space is simply the potential of its pre-image under F.

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Note that if we are interested in potential-change we can ignore the additive term above. Then f and $\hat{f}$ have the same form; only c is replaced by $\hat{c}$.

The basic idea is to show that one iteration of Karmarkar results in a constant decrease of $\hat{f}$. This means

$$\hat{f}(\hat{x}) \leq \hat{f}(\frac{e}{n}) - \delta \ ,$$

where δ is a constant.

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This gives

$$f(\bar{x}_{\text{new}}) \leq f(\bar{x}) - \delta \ .$$

Lemma 3

There is a feasible point z (i.e., $\hat{A}z = 0$) in $B(\frac{e}{n}, \rho) \cap \Delta$ that has

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with $\delta = \ln(1+\alpha)$.

Note that this shows the existence of a good point within the ball. In general it will be difficult to find this point.

Let z^* be the feasible point in the transformed space where $\hat{c}^t x$ is minimized. (Note that in contrast $\hat{x}$ is the point in the intersection of the feasible region and $B(\frac{e}{n}, \rho)$ that minimizes this function; in general $z^* \neq \hat{x}$)

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z^* must lie at the boundary of the simplex. This means $z^* \notin B(\frac{e}{n}, \rho)$.

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The point z we want to use lies farthest in the direction from $\frac{e}{n}$ to z^*, namely

$$z = (1 - \lambda)\frac{e}{n} + \lambda z^*$$

for some positive $\lambda < 1$.

Hence,

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Therefore,

$$\frac{\hat{c}^t \frac{e}{n}}{\hat{c}^t z} = \frac{1}{1 - \lambda}$$

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&= \sum_j \ln(\frac{n}{1-\lambda}((1-\lambda)\frac{1}{n} + \lambda z_j^*)) \\
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$$1 + n\frac{\lambda}{1-\lambda} \ge 1 + \frac{n\alpha}{n-\alpha-1} \ge 1 + \alpha$$

This gives the lemma.

Lemma 4

If we choose $\alpha = 1/4$ *and* $n \geq 4$ *in Karmarkars algorithm the point* $\hat{x}$ *satisfies*

$$\hat{f}(\hat{x}) \leq \hat{f}(\frac{e}{n}) - \delta$$

with $\delta = 1/10$.

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Proof:

Define

$$g(x) = n\ln\frac{\hat{c}^t x}{\hat{c}^t\frac{e}{n}} = n(\ln\hat{c}^t x - \ln\hat{c}^t\frac{e}{n}) \ .$$

This is the change in the cost part of the potential function when going from the center $\frac{e}{n}$ to the point x in the transformed space.

Similar, the penalty when going from $\frac{e}{n}$ to w increases by

$$h(w) = \mathrm{pen}(w) - \mathrm{pen}(\frac{e}{n}) = -\sum_j \ln \frac{w_j}{\frac{1}{n}}$$

where $\mathrm{pen}(v) = -\sum_j \ln(v_j)$.

We want to derive a lower bound on

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$$\begin{aligned}\hat{f}(\frac{e}{n}) - \hat{f}(\hat{x}) = {} & [\hat{f}(\frac{e}{n}) - \hat{f}(z)] \\ & + h(z) \\ & - h(x) \\ & + [g(z) - g(\hat{x})]\end{aligned}$$

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where z is the point in the ball where $\hat{f}$ achieves its minimum.

We have

$$[\hat{f}(\frac{e}{n}) - \hat{f}(z)] \geq \ln(1+\alpha)$$

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We have

$$[g(z) - g(\hat{x})] \geq 0$$

since $\hat{x}$ is the point with minimum cost in the ball, and g is monotonically increasing with cost.

For a point in the ball we have

$$\hat{f}(w) - (\hat{f}(\frac{e}{n}) + g(w))h(w)$$

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This is at most $\frac{\beta^2}{2(1-\beta)}$ with $\beta = n\alpha r$.

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This is at most $\frac{\beta^2}{2(1-\beta)}$ with $\beta = n\alpha r$.
Hence,

$$\hat{f}(\frac{e}{n}) - \hat{f}(\hat{x}) \geq \ln(1+\alpha) - \frac{\beta^2}{(1-\beta)} \quad .$$

Lemma 5
For $|x| \leq \beta < 1$

$$|\ln(1+x) - x| \leq \frac{x^2}{2(1-\beta)} \enspace .$$

This gives for $w \in B(\frac{e}{n}, \rho)$

$$\left| \sum_j \ln \frac{w_j}{1/n} \right|$$

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This gives for $w \in B(\frac{e}{n}, \rho)$

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&= \left|\sum_j \left[\ln(1 + \overbrace{n(w_j - 1/n)}^{\leq n\alpha r < 1}) - n(w_j - \frac{1}{n})\right]\right| \\
&\leq \sum_j \frac{n^2(w_j - 1/n)^2}{2(1 - \alpha n r)}
\end{aligned}
$$

This gives for $w \in B(\frac{e}{n}, \rho)$

$$
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\left|\sum_j \ln \frac{w_j}{1/n}\right| &= \left|\sum_j \ln(\frac{1/n + (w_j - 1/n)}{1/n}) - \sum_j n(w_j - \frac{1}{n})\right| \\
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&\le \sum_j \frac{n^2(w_j - 1/n)^2}{2(1 - \alpha n r)} \\
&\le \frac{(\alpha n r)^2}{2(1 - \alpha n r)}
\end{aligned}
$$

The decrease in potential is therefore at least

$$\ln(1+\alpha) - \frac{\beta^2}{1-\beta}$$

with $\beta = n\alpha r = \alpha\sqrt{\frac{n}{n-1}}$.

It can be shown that this is at least $\frac{1}{10}$ for $n \geq 4$ and $\alpha = 1/4$.

The decrease in potential is therefore at least

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Let $\bar{x}^{(k)}$ be the current point after the k-th iteration, and let $\bar{x}^{(0)} = \frac{e}{n}$.

Then $f(\bar{x}^{(k)}) \leq f(e/n) - k/10$.
This gives

$$n \ln \frac{c^t \bar{x}^{(k)}}{c^t \frac{e}{n}} \leq \sum_j \ln \bar{x}_j^{(k)} - \sum_j \ln \frac{1}{n} - k/10$$
$$\leq n \ln n - k/10$$

Choosing $k = 10n(\ell + \ln n)$ with $\ell = \Theta(L)$ we get

$$\frac{c^t \bar{x}^{(k)}}{c^t \frac{e}{n}} \leq e^{-\ell} \leq 2^{-\ell} .$$

Hence, $\Theta(nL)$ iterations are sufficient. One iteration can be performed in time $\mathcal{O}(n^3)$.

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