8 Seidels LP-algorithm

- Suppose we want to solve $\min\{c^t x \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- If d is much smaller than m one can do a lot better.
- In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., linear in m.

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Setting:

- We assume an LP of the form

$$\begin{array}{llll} \min & c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

- Further we assume that the LP is non-degenerate.
- We assume that the optimum solution is unique.
- We assume that the LP is bounded.

Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{lrcl} \min & c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^t x$ for any basic feasible solution.**

Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b.

Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables; denote the resulting matrix with $\bar{A}$.

If B is an optimal basis then x_B with $\bar{A}_B x_B = b$, gives an optimal assignment to the basis variables (non-basic variables are 0).

Theorem 2 (Cramers Rule)

Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_j = \frac{\det(M_j)}{\det(M)} \ ,$$

where M_j is the matrix obtained from M by replacing the j-th column by the vector b.

Proof:

- Define

$$X_j = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{j-1} & x & e_{j+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$

  Note that expanding along the j-th column gives that $\det(X_j) = x_j$.

- Further, we have

$$MX_j = \begin{pmatrix} | & & | & | & | & & | \\ Me_1 & \cdots & Me_{j-1} & Mx & Me_{j+1} & \cdots & Me_n \\ | & & | & | & | & & | \end{pmatrix} = M_j$$

- Hence,

$$x_j = \det(X_j) = \frac{\det(M_j)}{\det(M)}$$

Bounding the Determinant

Let Z be the maximum absolute entry occuring in A, b or c. Let C denote the matrix obtained from $\bar{A}_B$ by replacing the j-th column with vector b.

Observe that

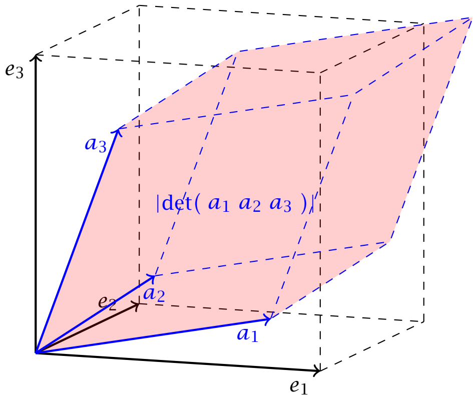
$$
\begin{aligned}
|\det(C)| &= \left| \sum_{\pi \in S_m} \prod_{1 \le i \le m} \operatorname{sgn}(\pi) C_{i\pi(i)} \right| \\
&\le \sum_{\pi \in S_m} \prod_{1 \le i \le m} |C_{i\pi(i)}| \\
&\le m! \cdot Z^m \enspace .
\end{aligned}
$$

Bounding the Determinant

Alternatively, Hadamards inequality gives

$$|\det(C)| \leq \prod_{i=1}^{m} \|C_{*i}\| \leq \prod_{i=1}^{m} (\sqrt{m}Z)$$
$$\leq m^{m/2} Z^m \quad .$$

Hadamards Inequality



Hadamards inequality says that the red volume is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{rrcl} \min & c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^t x$ for any basic feasible solution.** Add the constraint $c^t x \geq -mZ(m! \cdot Z^m) - 1$. Note that this constraint is superfluous unless the LP is unbounded.

Ensuring Conditions

Make the LP non-degenerate by perturbing the right-hand side vector b.

Make the LP solution unique by perturbing the optimization direction c.

Compute an optimum basis for the new LP.

- If the cost is $c^t x = -(mZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
- Otw. we have an optimum basis.

In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^t x \geq -mZ(m! \cdot Z^m) - 1$.

We give a routine SeidelLP$(\mathcal{H}, d)$ that is given a set $\mathcal{H}$ of explicit, non-degenerate constraints over d variables, and minimizes $c^t x$ over all feasible points.

In addition it obeys the implicit constraint
$c^t x \geq -(mZ)(m! \cdot Z^m) - 1$.

Algorithm 1 SeidelLP($\mathcal{H}, d$)

if $d = 1$ **then** solve 1-dimensional problem and return;
if $\mathcal{H} = \emptyset$ **then** return x on implicit constraint hyperplane
choose random constraint $h \in \mathcal{H}$
$\hat{\mathcal{H}} \leftarrow \mathcal{H} \setminus \{h\}$
$\hat{x}^* \leftarrow$ SeidelLP($\hat{\mathcal{H}}, d$)
if $\hat{x}^* =$ infeasible **then return** infeasible
if $\hat{x}^*$ fulfills h **then return** $\hat{x}^*$
// optimal solution fulfills h with equality, i.e., $A_h x = b_h$
solve $A_h x = b_h$ for some variable x_ℓ;
eliminate x_ℓ in constraints from $\hat{\mathcal{H}}$ and in implicit constr.;
$\hat{x}^* \leftarrow$ SeidelLP($\hat{\mathcal{H}}, d - 1$)
if $\hat{x}^* =$ infeasible **then**
    return infeasible
else
    add the value of x_ℓ to $\hat{x}^*$ and return the solution

8 Seidels LP-algorithm

- If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(m)$.
- If $d > 1$ and $m = 0$ we take time $\mathcal{O}(d)$ to return d-dimensional vector x.
- The first recursive call takes time $T(m-1, d)$ for the call plus $\mathcal{O}(d)$ for checking whether the solution fulfills h.
- If we are unlucky and $\hat{x}^*$ does not fulfill h we need time $\mathcal{O}(d(m+1)) = \mathcal{O}(dm)$ to eliminate x_ℓ. Then we make a recursive call that takes time $T(m-1, d-1)$.
- The probability of being unlucky is at most d/m as there are at most d constraints whose removal will decrease the objective function (recall that the solution is unique).

8 Seidels LP-algorithm

This gives the recurrence

$$T(m,d) = \begin{cases} \mathcal{O}(m) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1,d) + & \\ \frac{d}{m}(\mathcal{O}(dm) + T(m-1,d-1)) & \text{otw.} \end{cases}$$

Note that $T(m,d)$ denotes the expected running time.

8 Seidels LP-algorithm

Let C be the largest constant in the $\mathcal{O}$-notations.

We show $T(m,d) \le Cf(d)\max\{1,m\}$.

$d = 1$:

$$T(m,1) \le Cm \le Cf(1)\max\{1,m\} \text{ for } f(1) \ge 1$$

$d > 1; m = 0$:

$$T(0,d) \le \mathcal{O}(d) \le Cd \le Cf(d)\max\{1,m\} \text{ for } f(d) \ge d$$

$d > 1; m = 1$:

$$\begin{aligned}T(1,d) &= \mathcal{O}(d) + T(0,d) + d\Big(\mathcal{O}(d) + T(0,d-1)\Big)\\ &\le Cd + Cd + Cd^2 + dT(0,d-1)\\ &\le Cf(d)\max\{1,m\} \text{ for } f(d) \ge 4d^2\end{aligned}$$

8 Seidels LP-algorithm

$\boldsymbol{d > 1; m > 1}$ **:**
(by induction hypothesis statm. true for $d' < d, m' \geq 0$;
and for $d' = d$, $m' < m$)

$$
\begin{aligned}
T(m,d) &= \mathcal{O}(d) + T(m-1,d) + \frac{d}{m}\Big(\mathcal{O}(dm) + T(m-1,d-1)\Big) \\
&\leq Cd + Cf(d)(m-1) + Cd^2 + \frac{d}{m}Cf(d-1)(m-1) \\
&\leq 2Cd^2 + Cf(d)(m-1) + dCf(d-1) \\
&\leq Cf(d)m
\end{aligned}
$$

if $f(d) \geq df(d-1) + 2d^2$.

8 Seidels LP-algorithm

- Define $f(1) = 4 \cdot 1^2$ and $f(d) = df(d-1) + 4d^2$ for $d > 1$.

Then

$$
\begin{aligned}
f(d) &= 4d^2 + df(d-1) \\
&= 4d^2 + d\left[4(d-1)^2 + (d-1)f(d-2)\right] \\
&= 4d^2 + d\left[4(d-1)^2 + (d-1)\left[4(d-2)^2 + (d-2)f(d-3)\right]\right] \\
&= 4d^2 + 4d(d-1)^2 + 4d(d-1)(d-2)^2 + \dots \\
&\quad + 4d(d-1)(d-2)\cdot \ldots \cdot 4 \cdot 3 \cdot 1^2 \\
&= 4d!\left(\frac{d^2}{d!} + \frac{(d-1)^2}{(d-1)!} + \frac{(d-2)^2}{(d-2)!} + \dots\right) \\
&= \mathcal{O}(d!)
\end{aligned}
$$

since $\sum_{i \geq 1} \frac{i^2}{i!}$ is a constant.