

Technique 4: The Greedy Algorithm

Algorithm 1 Greedy

- 1: $I \leftarrow \emptyset$
- 2: $\hat{S}_j \leftarrow S_j$ for all j
- 3: **while** I not a set cover **do**
- 4: $\ell \leftarrow \arg \min_{j: \hat{S}_j \neq \emptyset} \frac{w_j}{|\hat{S}_j|}$
- 5: $I \leftarrow I \cup \{\ell\}$
- 6: $\hat{S}_j \leftarrow \hat{S}_j - S_\ell$ for all j

In every round the Greedy algorithm takes the set that covers remaining elements in the most **cost-effective** way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.

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Lemma 4

Given positive numbers a_1, \dots, a_k and b_1, \dots, b_k , and $S \subseteq \{1, \dots, k\}$ then

$$\min_i \frac{a_i}{b_i} \leq \frac{\sum_{i \in S} a_i}{\sum_{i \in S} b_i} \leq \max_i \frac{a_i}{b_i}$$

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Let n_ℓ denote the number of elements that remain at the beginning of iteration ℓ . $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need s iterations.

In the ℓ -th iteration

$$\min_j \frac{w_j}{|\hat{S}_j|} \leq \frac{\sum_{j \in \text{OPT}} w_j}{\sum_{j \in \text{OPT}} |\hat{S}_j|} = \frac{\text{OPT}}{\sum_{j \in \text{OPT}} |\hat{S}_j|} \leq \frac{\text{OPT}}{n_\ell}$$

since an optimal algorithm can cover the remaining n_ℓ elements with cost OPT.

Let \hat{S}_j be a subset that minimizes this ratio. Hence,
 $w_j/|\hat{S}_j| \leq \frac{\text{OPT}}{n_\ell}$.

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Adding this set to our solution means $n_{\ell+1} = n_{\ell} - |\hat{S}_j|$.

$$w_j \leq \frac{|\hat{S}_j| \text{OPT}}{n_{\ell}} = \frac{n_{\ell} - n_{\ell+1}}{n_{\ell}} \cdot \text{OPT}$$

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$$\begin{aligned}\sum_{j \in I} w_j &\leq \sum_{\ell=1}^s \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \text{OPT} \\ &\leq \text{OPT} \sum_{\ell=1}^s \left(\frac{1}{n_\ell} + \frac{1}{n_\ell - 1} + \dots + \frac{1}{n_{\ell+1} + 1} \right) \\ &= \text{OPT} \sum_{i=1}^k \frac{1}{i} \\ &= H_n \cdot \text{OPT} \leq \text{OPT}(\ln n + 1) .\end{aligned}$$

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A tight example:

