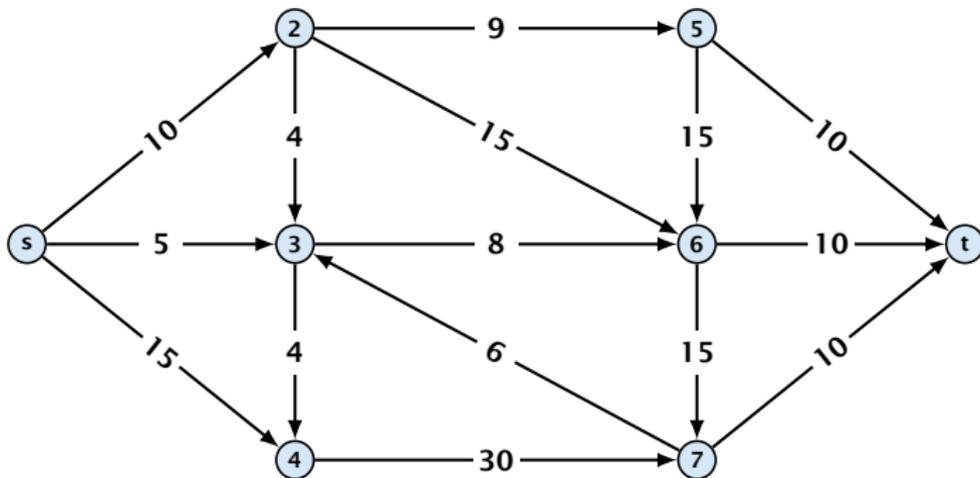# Part IV

Flows and Cuts

11 Introduction

Flow Network

- ▶ directed graph $G = (V, E)$; edge capacities $c(e)$
- ▶ two special nodes: source s; target t;
- ▶ no edges entering s or leaving t;
- ▶ at least for now: no parallel edges;
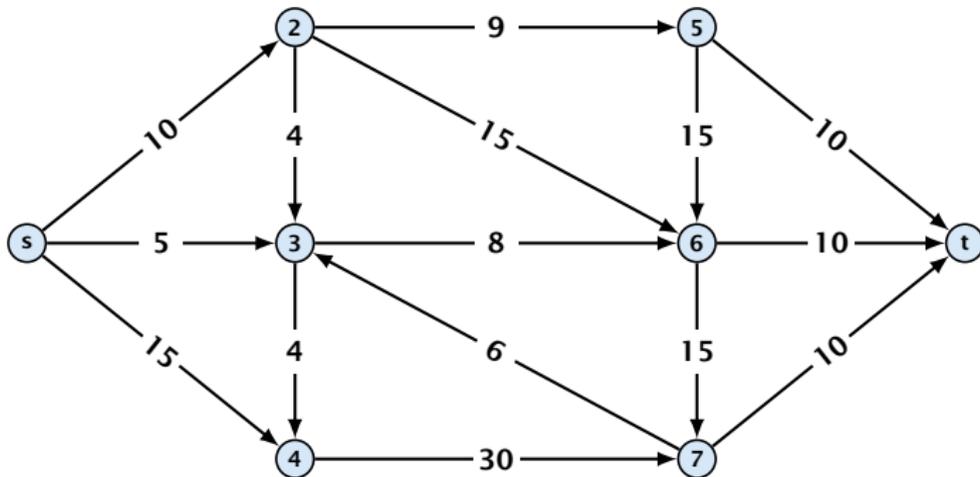
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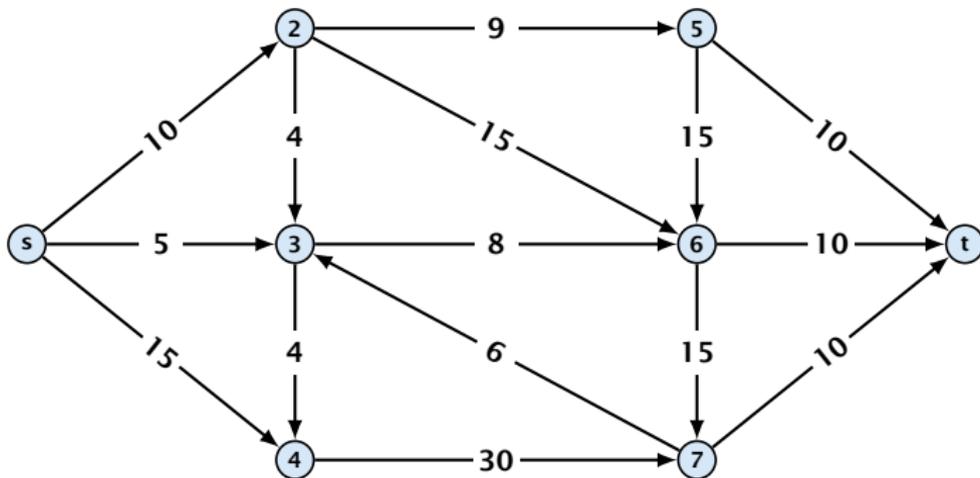# 11 Introduction

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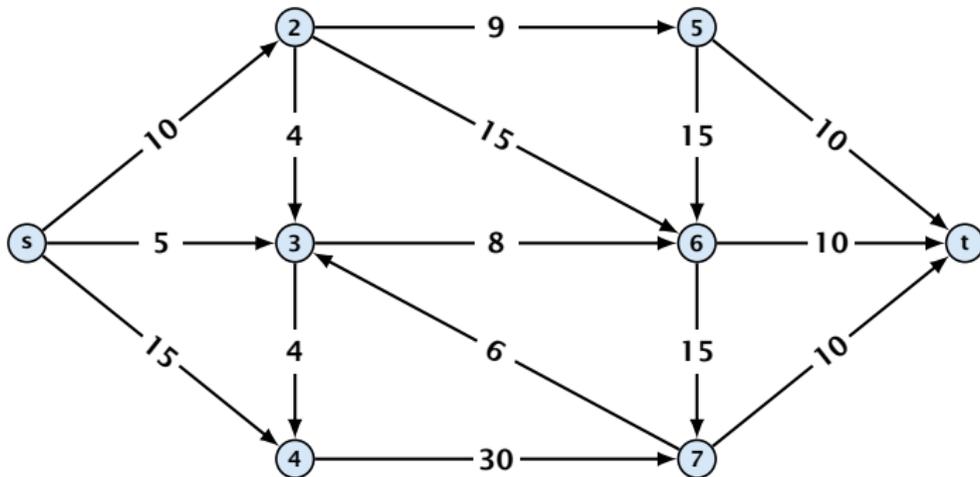
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$$\text{cap}(A, V \setminus A) := \sum_{e \,\in\, \text{out}(A)} c(e) \ ,$$

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Minimum Cut Problem: Find an (s, t)-cut with minimum capacity.

Cuts

Example 3
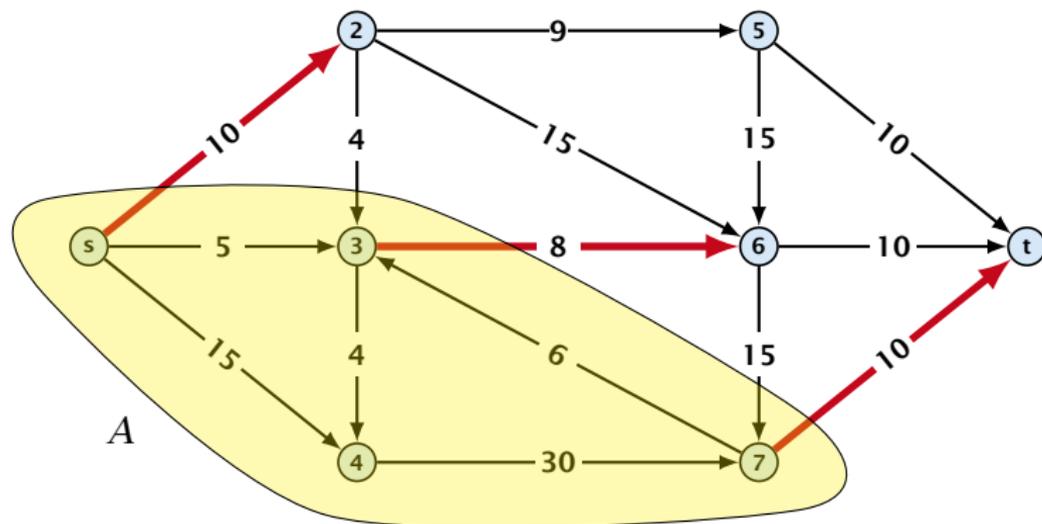
The capacity of the cut is $\text{cap}(A, V \setminus A) = 28$.

Flows

Definition 4

An (s, t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \ .$$

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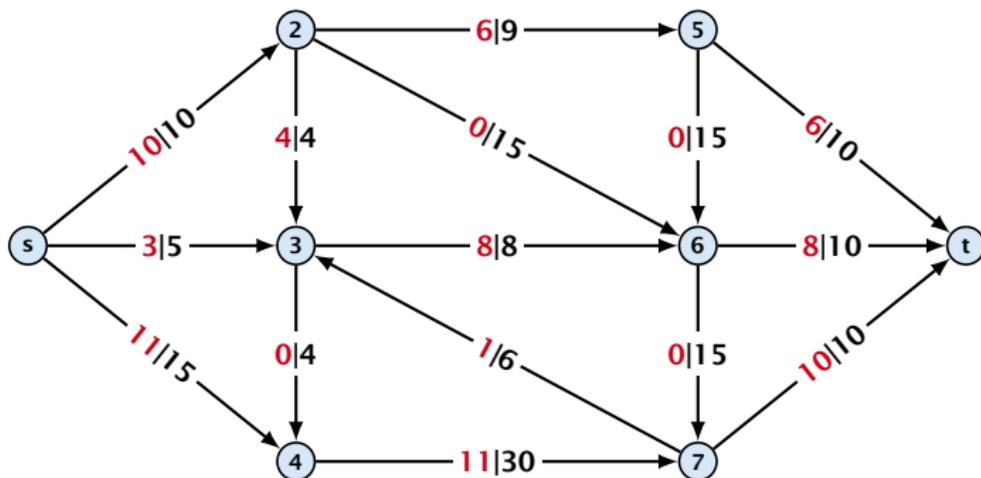
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Flows

Example 6

The value of the flow is $\text{val}(f) = 24$.

Flows

Lemma 7 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \ .$$

Proof.

$\text{val}(f)$

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$$= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A. $\qquad \square$

Example 8
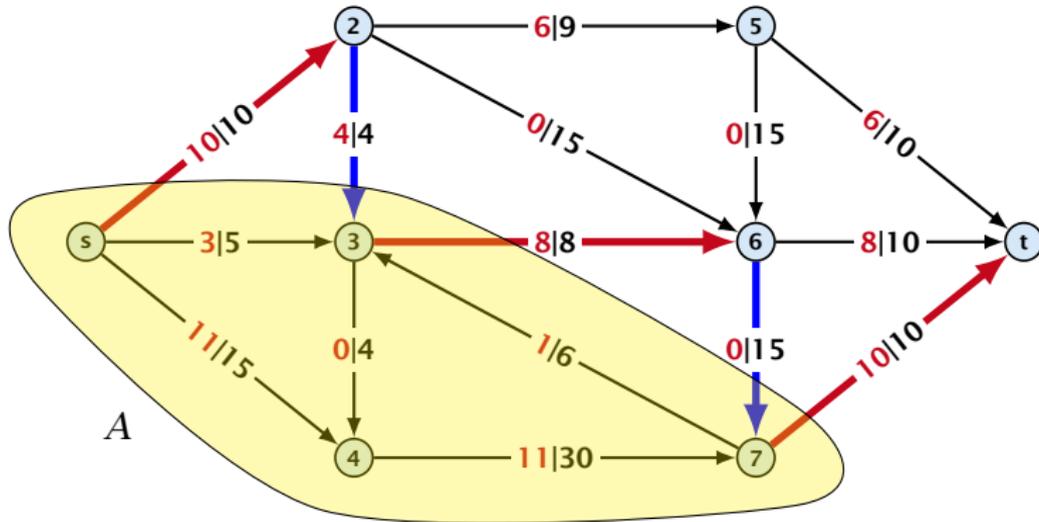
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12 Augmenting Path Algorithms

Greedy-algorithm:

▶ start with $f(e) = 0$ everywhere

▶ find an s-t path with $f(e) < c(e)$ on every edge

▶ augment flow along the path

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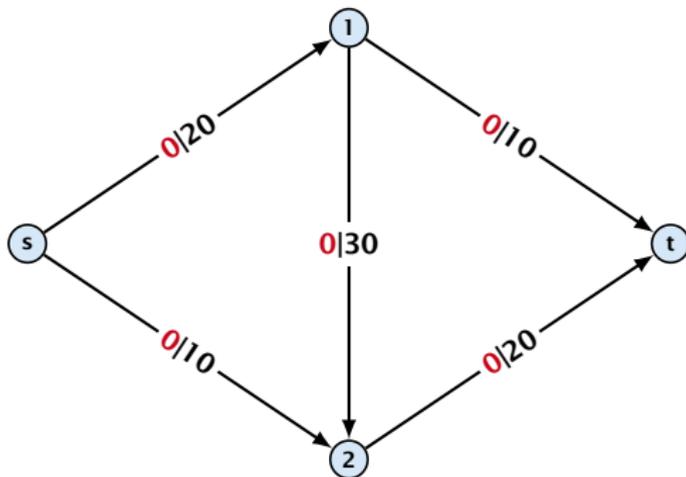
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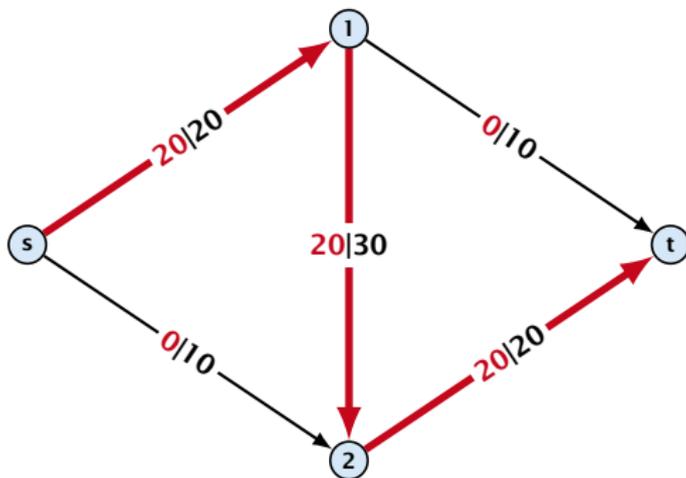
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Augmenting Path Algorithm

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An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 46 FordFulkerson($G = (V, E, c)$)
1: Initialize $f(e) \leftarrow 0$ for all edges.
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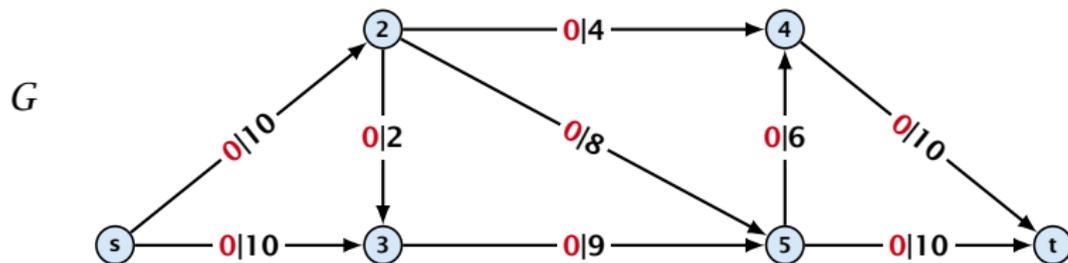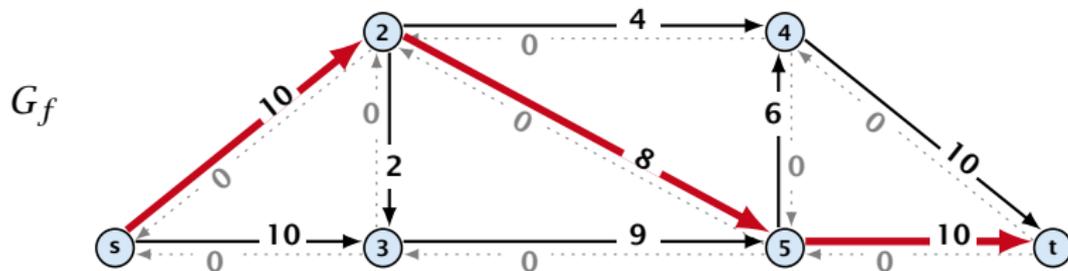3:   augment as much flow along p as possible.
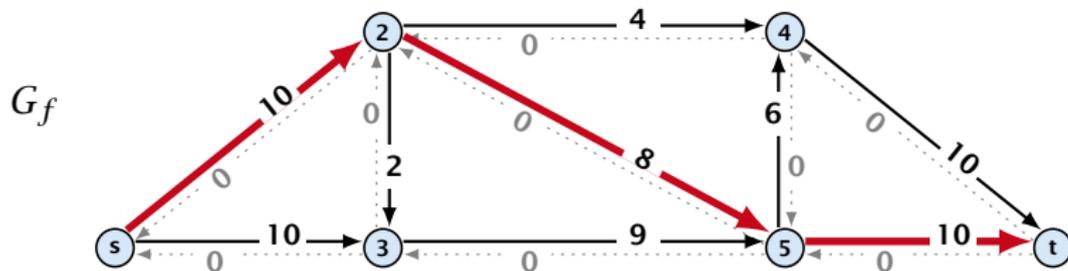

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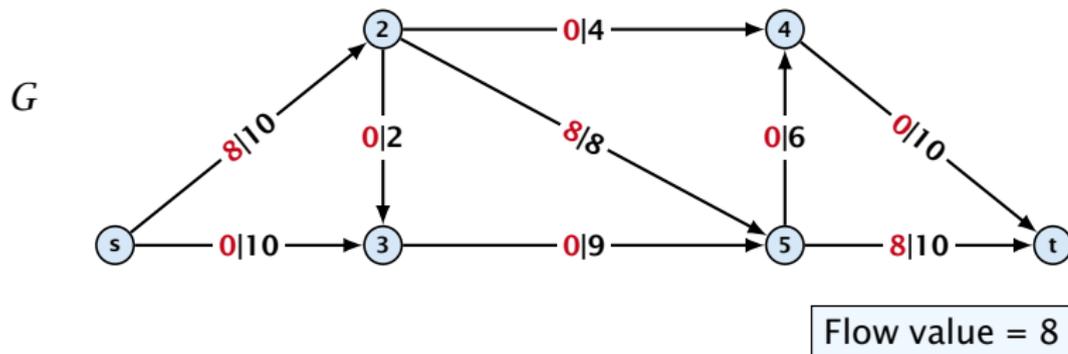3:   augment as much flow along p as possible.

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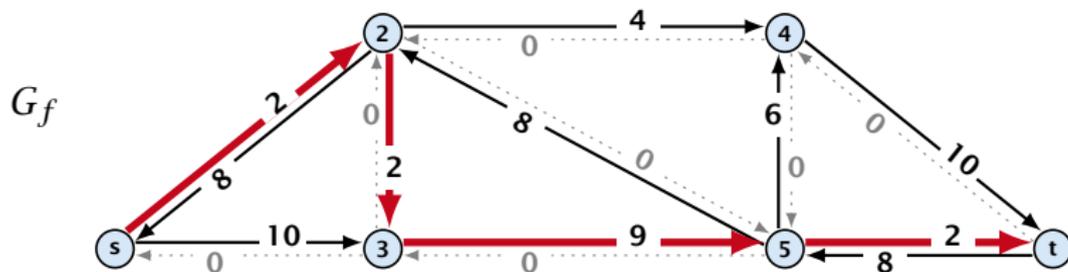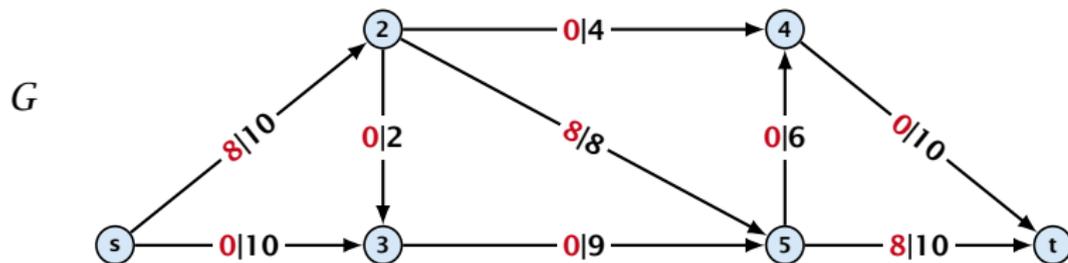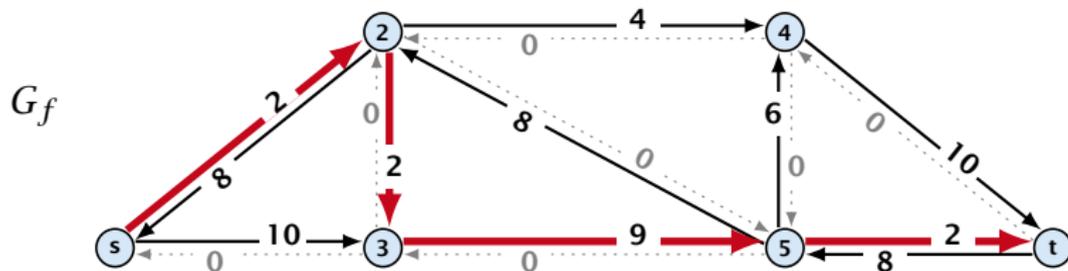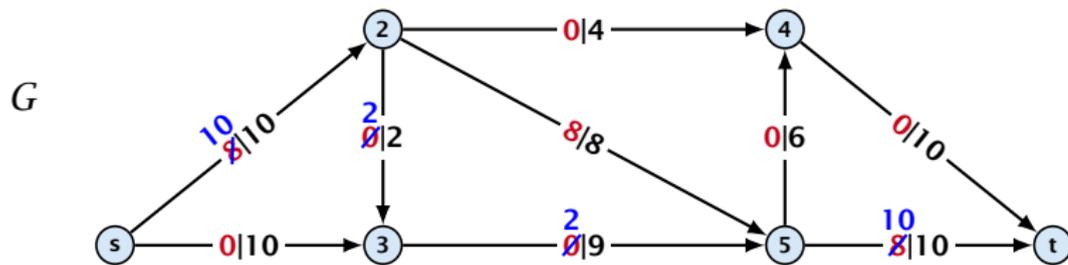

Flow value = 0

Augmenting Path Algorithm

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12.1 The Generic Augmenting Path Algorithm

Augmenting Path Algorithm

Flow value = 8

Augmenting Path Algorithm

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Augmenting Path Algorithm

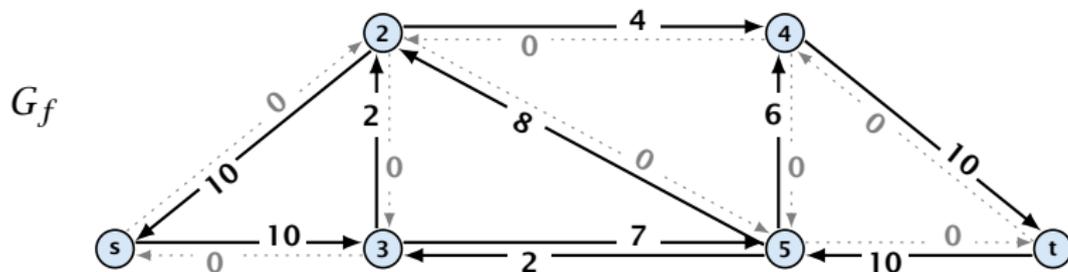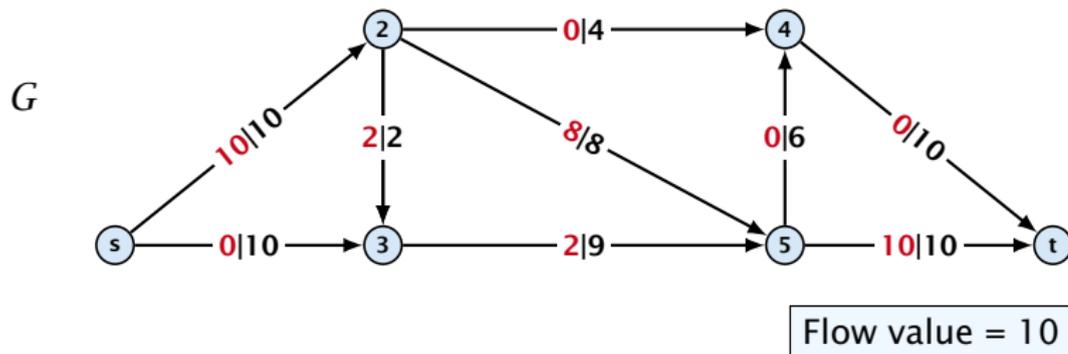Flow value = 10

Augmenting Path Algorithm

Flow value = 16
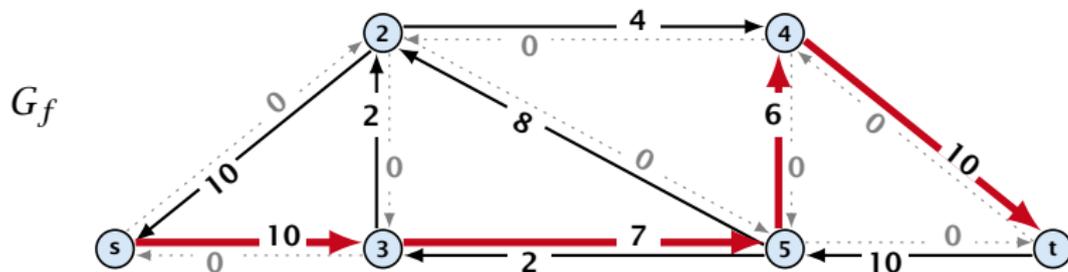
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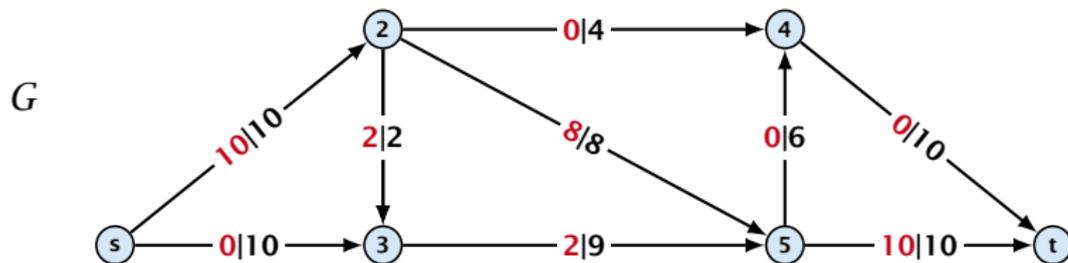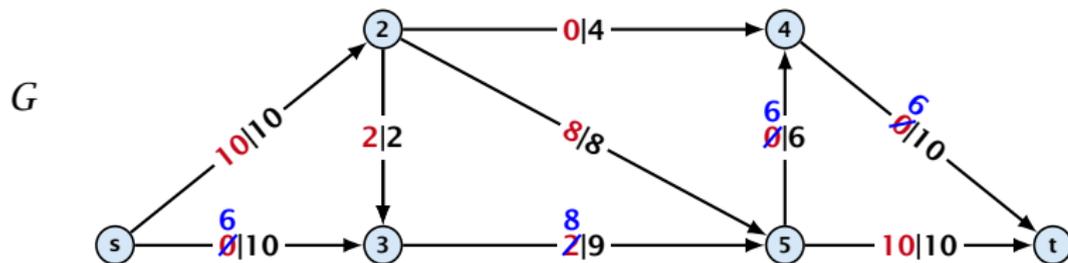
Flow value = 16

G_f

Augmenting Path Algorithm

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Augmenting Path Algorithm

Flow value = 18

Augmenting Path Algorithm

G

Flow value = 18

G_f

Augmenting Path Algorithm

Flow value = 18

Augmenting Path Algorithm

Flow value = 19

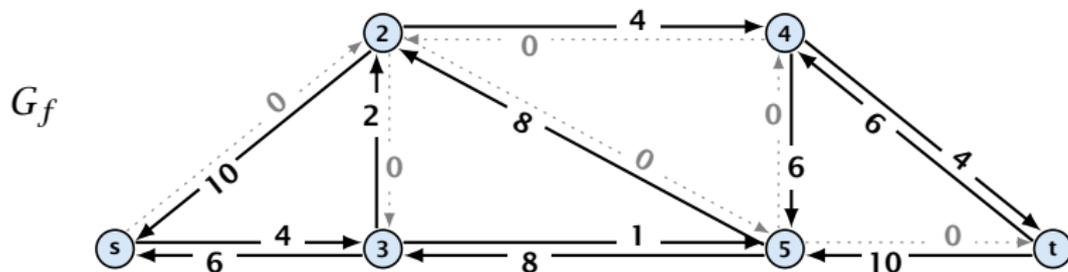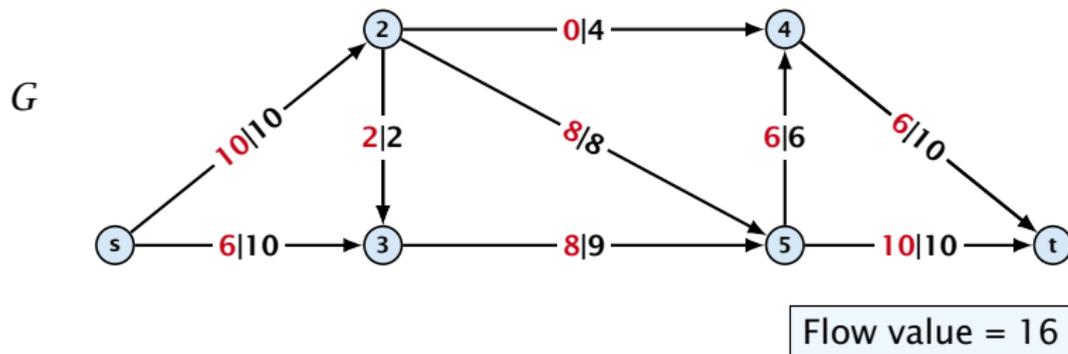# Augmenting Path Algorithm

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Augmenting Path Algorithm

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A flow f is a maximum flow iff there are no augmenting paths.

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The value of a maximum flow is equal to the value of a minimum cut.

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This we already showed.

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If there were an augmenting path, we could improve the flow. Contradiction.

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Augmenting Path Algorithm

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

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Assumption:

All capacities are integers between 1 and C.

Invariant:

Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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The algorithm terminates in at most $\operatorname{val}(f^*) \leq nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$. *This gives a total running time of* $\mathcal{O}(nmC)$.

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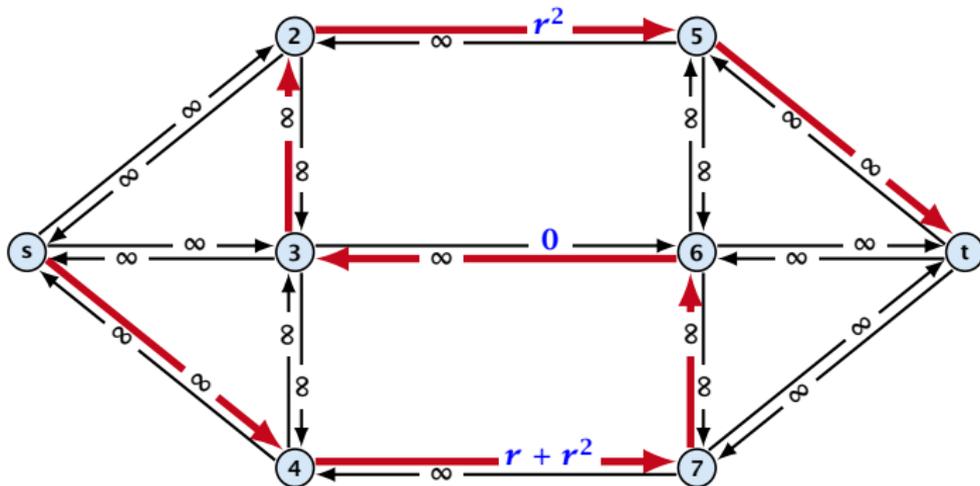
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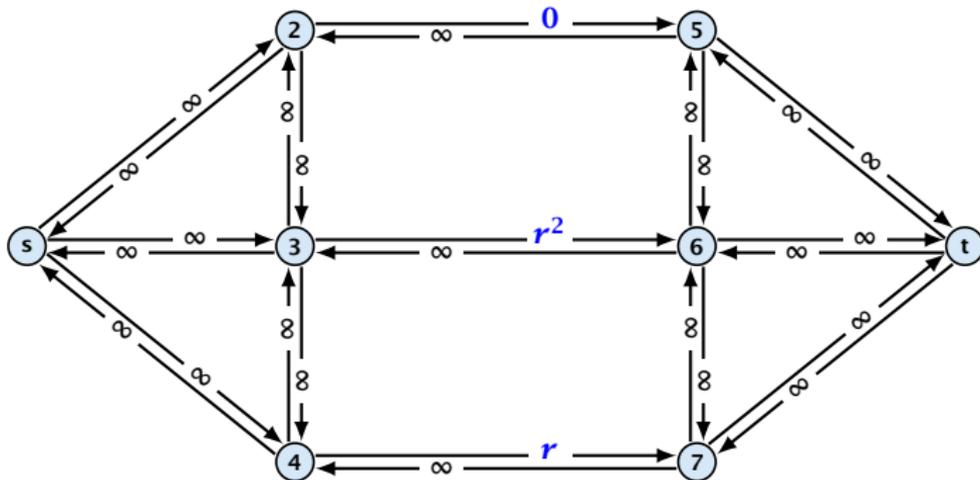
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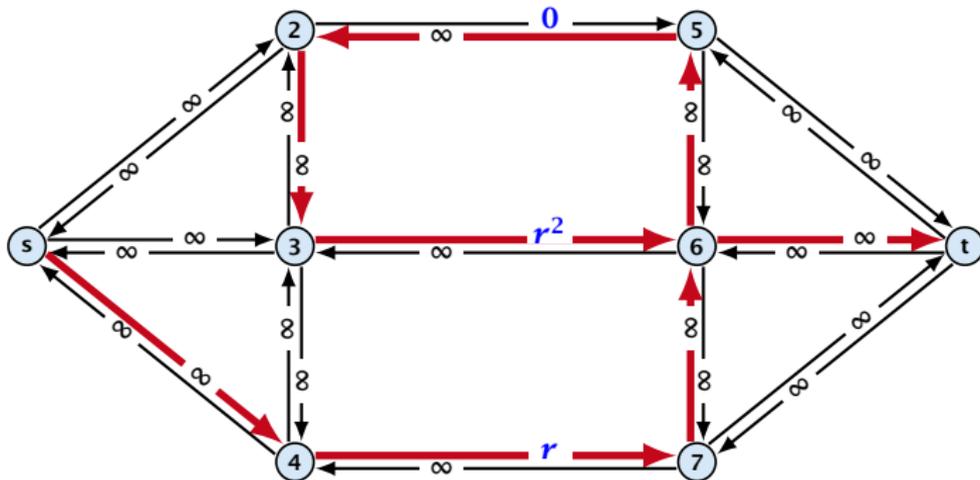
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Overview: Shortest Augmenting Paths

Lemma 15

The length of the shortest augmenting path never decreases.

Lemma 16

After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

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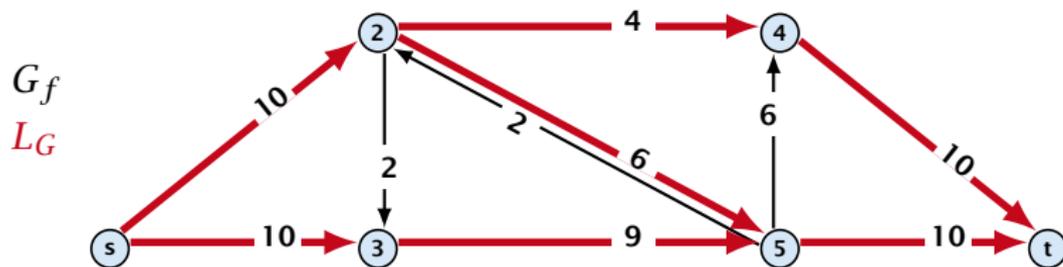
In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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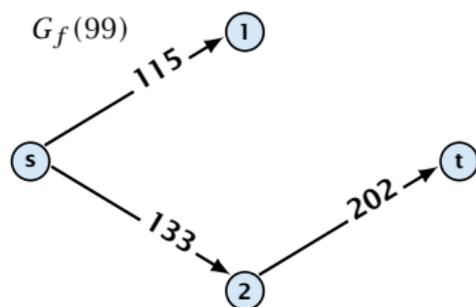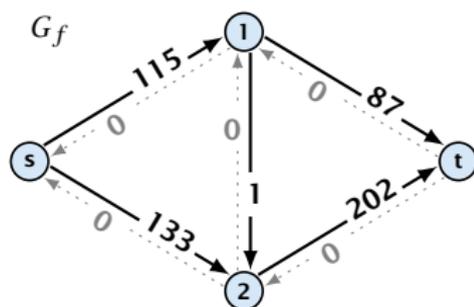
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Algorithm 2 maxflow(G, s, t, c)
1: **foreach** $e \in E$ **do** $f_e \leftarrow 0$;
2: $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
3: **while** $\Delta \geq 1$ **do**
4:     $G_f(\Delta) \leftarrow \Delta$-residual graph
5:         **while** there is augmenting path P in $G_f(\Delta)$ **do**
6:             $f \leftarrow$ augment(f, c, P)
7:             update$(G_f(\Delta))$
8:     $\Delta \leftarrow \Delta/2$
9: **return** f

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▶ This gives me an upper bound on the flow that I can still add.

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We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

Matching

- ▶ Input: undirected graph $G = (V, E)$.
- ▶ $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- ▶ Maximum Matching: find a matching of maximum cardinality
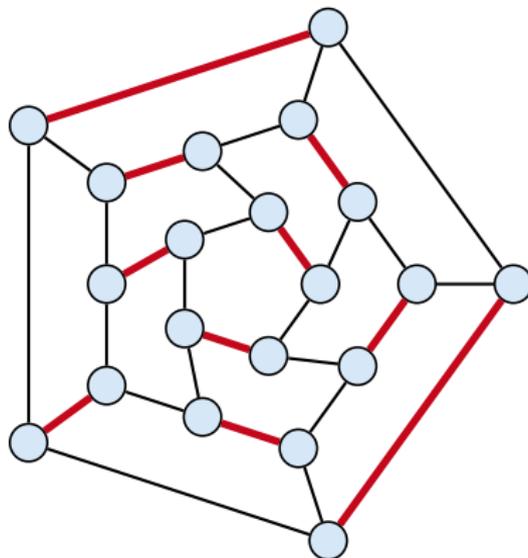
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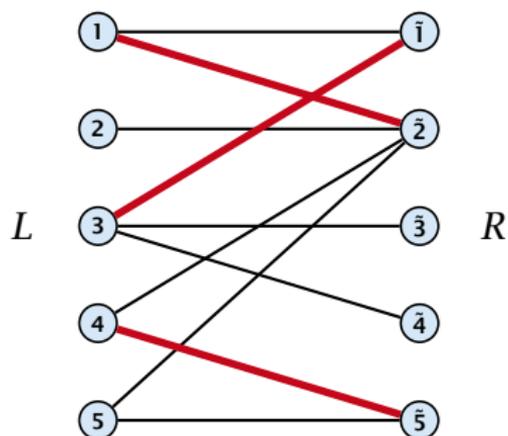
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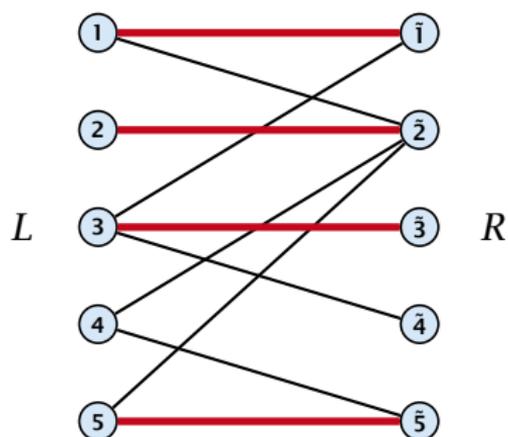
Maxflow Formulation

▶ Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s, t\}, E')$.
▶ Direct all edges from L to R.
▶ Add source s and connect it to all nodes on the left.
▶ Add t and connect all nodes on the right to t.
▶ All edges have unit capacity.

Proof

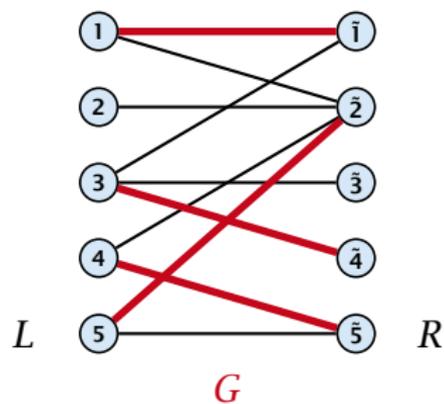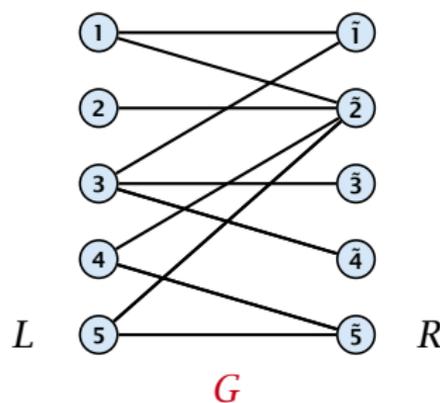**Max cardinality matching in $G \leq$ value of maxflow in G'**

- ▶ Given a maximum matching M of cardinality k.
- ▶ Consider flow f that sends one unit along each of k paths.
- ▶ f is a flow and has cardinality k.

Proof

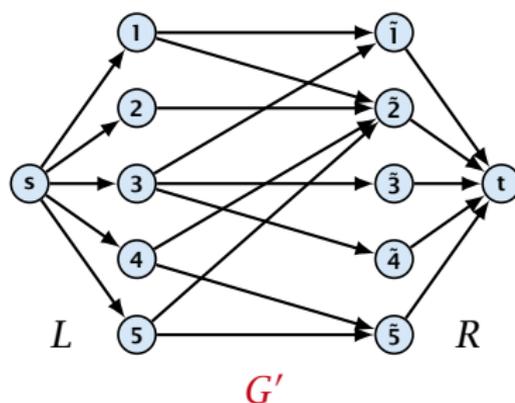**Max cardinality matching in G ≤ value of maxflow in G'**

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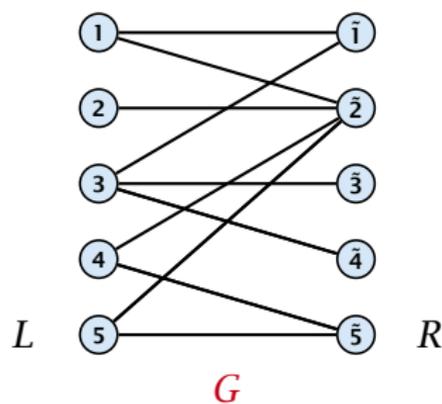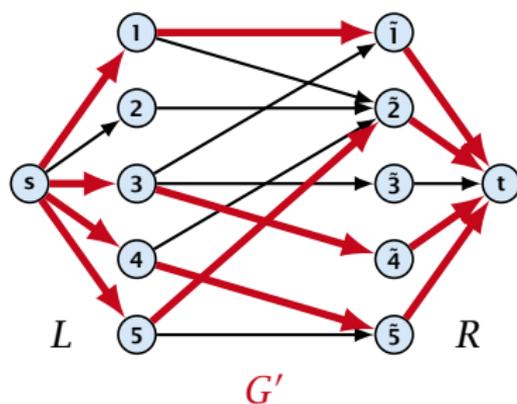**Max cardinality matching in $G \geq$ value of maxflow in G'**
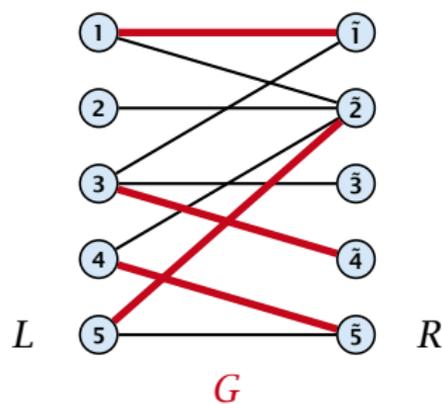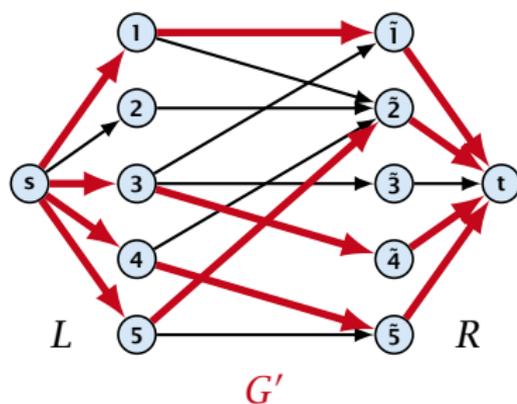
- ▶ Let f be a maxflow in G' of value k
- ▶ Integrality theorem $\Rightarrow$ k integral; we can assume f is 0/1.
- ▶ Consider $M=$ set of edges from L to R with $f(e) = 1$.
- ▶ Each node in L and R participates in at most one edge in M.
- ▶ $|M| = k$, as the flow must use at least k middle edges.

L G' R L G R

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13.1 Matching

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\operatorname{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

Baseball Elimination

team i	wins w_i	losses ℓ_i	remaining games			
			Atl	Phi	NY	Mon
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
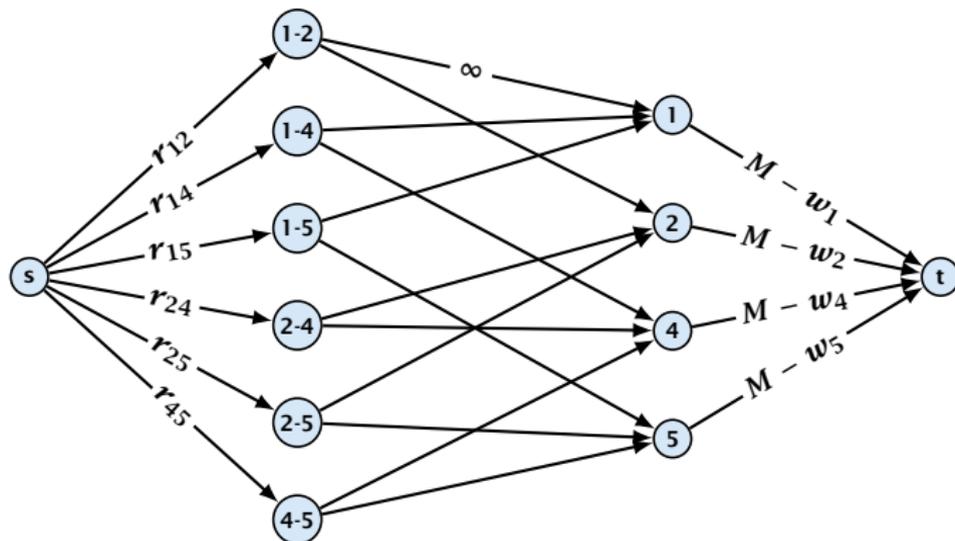
- ▶ But also Philadelphia is eliminated. Why?

Baseball Elimination

Formal definition of the problem:

- ▶ Given a set S of teams, and one specific team $z \in S$.
- ▶ Team x has already won w_x games.
- ▶ Team x still has to play team y, r_{xy} times.
- ▶ Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.

Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \qquad r(T) := \sum_{i,j \in T, i < j} r_{ij}$$

wins of teams in T

remaining games among teams in T

If $\frac{w(T) + r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 24

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

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$$r(S \setminus \{z\})$$

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$$r(S \setminus \{z\}) > \mathrm{cap}(A, V \setminus A)$$
$$\geq \sum_{i < j : i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i)$$

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- This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (⇒)

▶ Suppose we have a flow that saturates all source edges.

▶ We can assume that this flow is integral.

▶ For every pairing x-y it defines how many games team x and team y should win.

▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.

▶ This is less than $M - w_x$ because of capacity constraints.

▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.

▶ Hence, team z is not eliminated.

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Project Selection

Project selection problem:

▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

▶ Some projects have requirements (taking course EA2 requires course EA1).

▶ Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."

▶ A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

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The prerequisite graph:

- ▶ $\{x, a, z\}$ is a feasible subset.
- ▶ $\{x, a\}$ is infeasible.
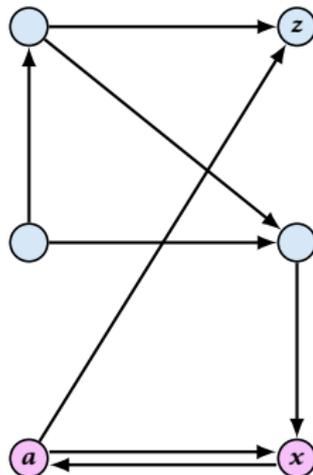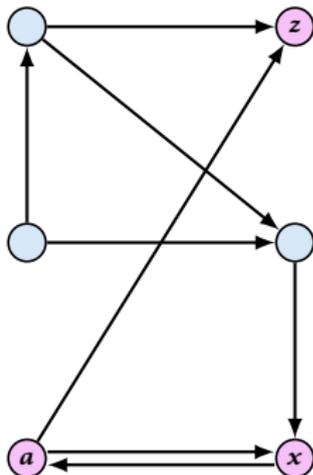
Project Selection

Mincut formulation:

- ▶ Edges in the prerequisite graph get infinite capacity.
- ▶ Add edge (s, v) with capacity p_v for nodes v with positive profit.
- ▶ Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.

prerequisite graph
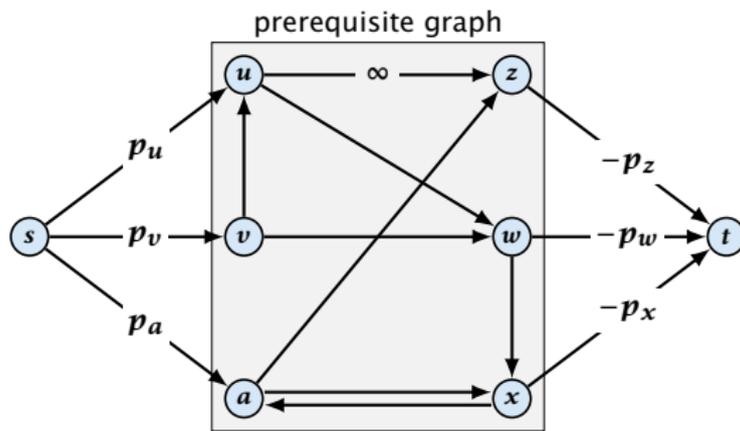
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▶ A is feasible because of capacity infinity edges.

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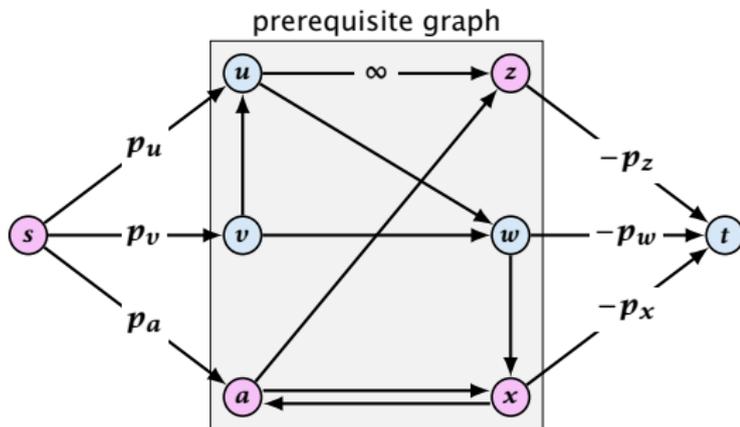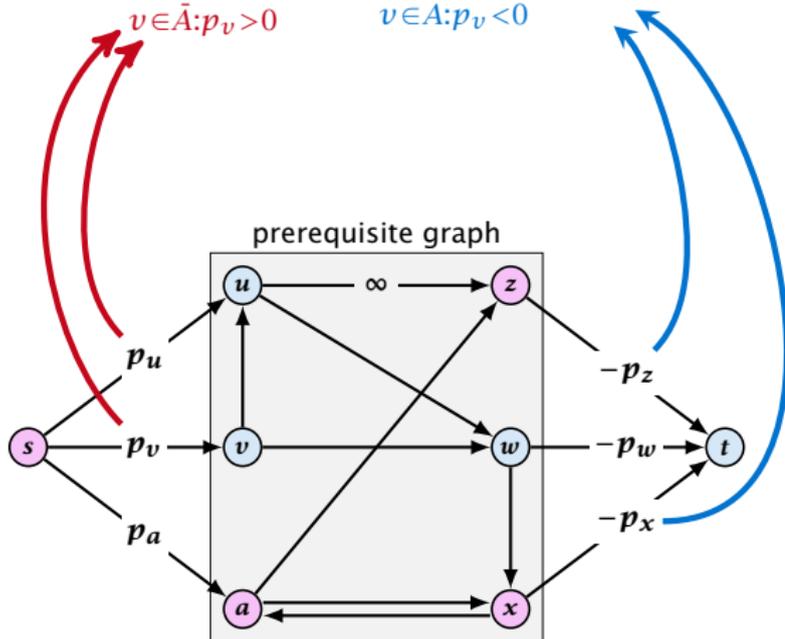
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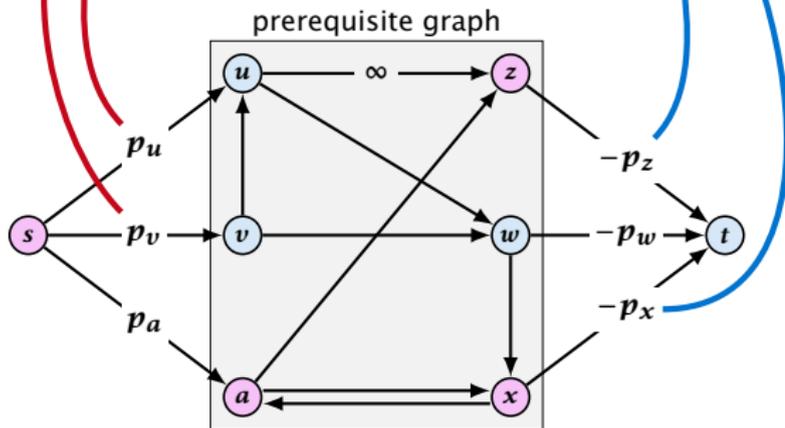
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Preflows

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1. For each edge e

$$0 \le f(e) \le c(e) \ .$$

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) \le \sum_{e \in \text{into}(v)} f(e) \ .$$

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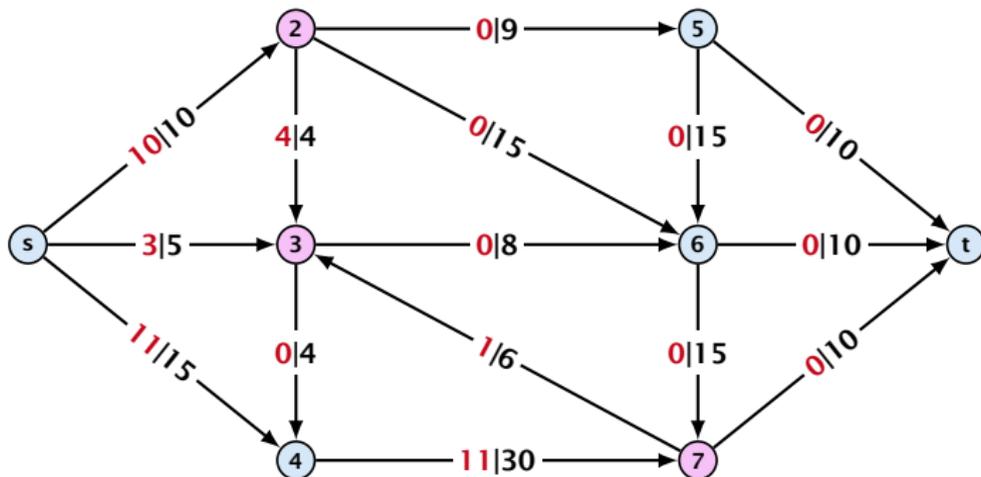
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Example 27
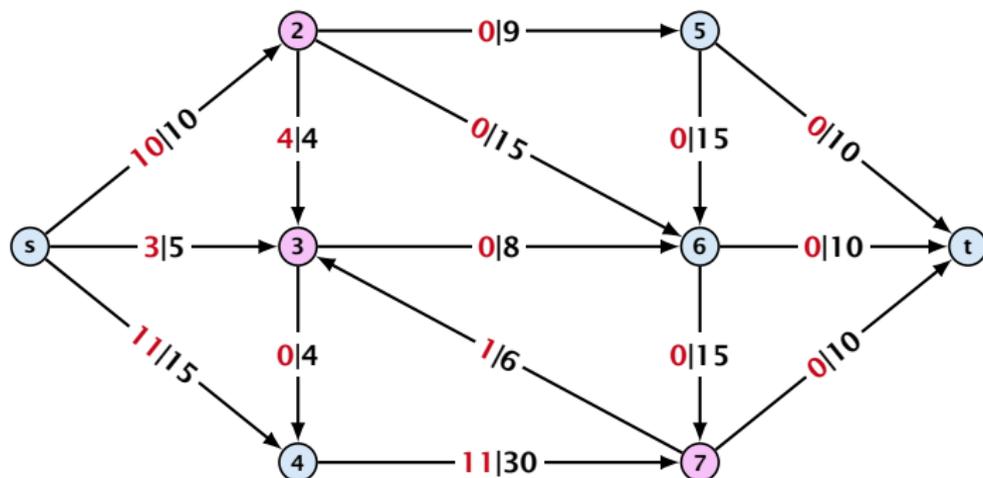
A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

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Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \leq \ell(v) + 1$ for all edges in the residual graph G_f (only non-zero capacity edges!!!)

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Intuition:
The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows
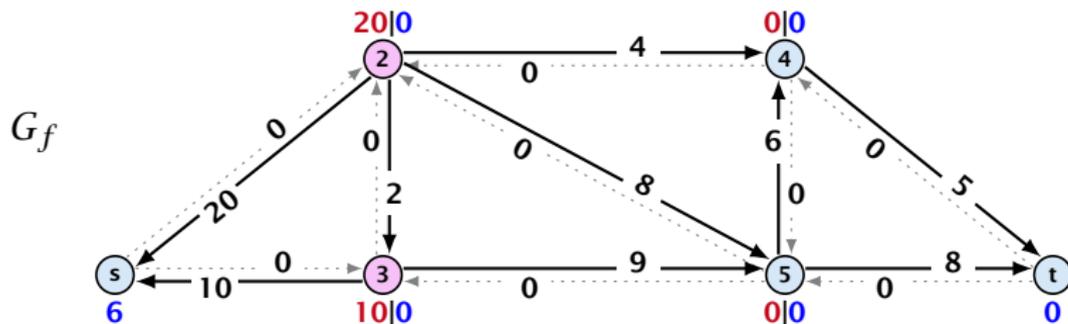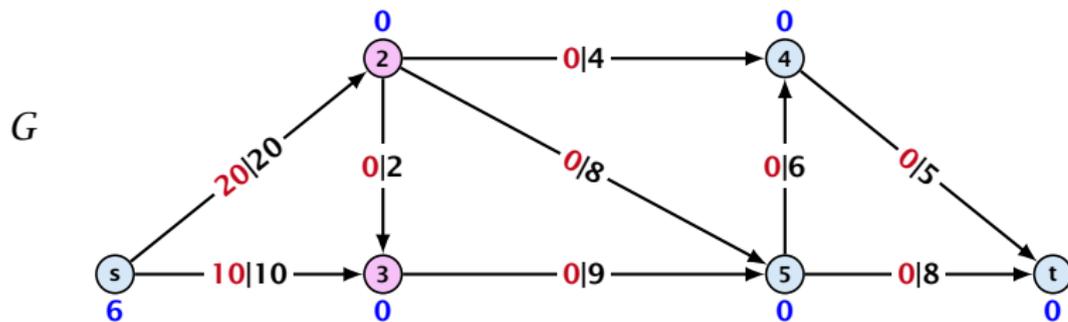
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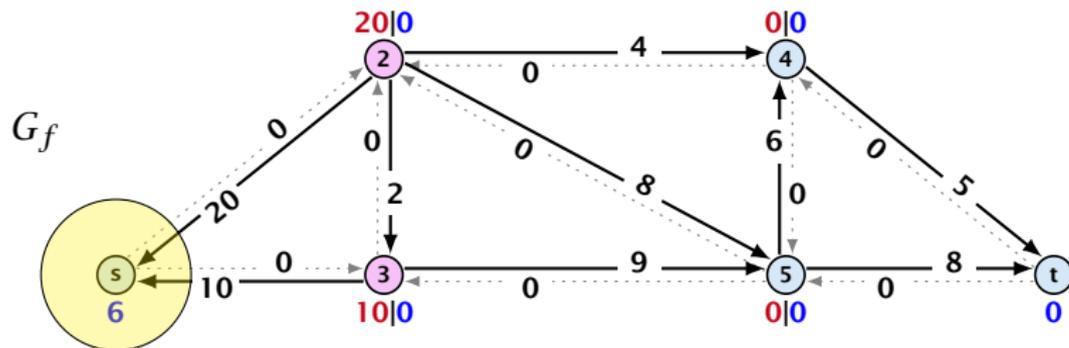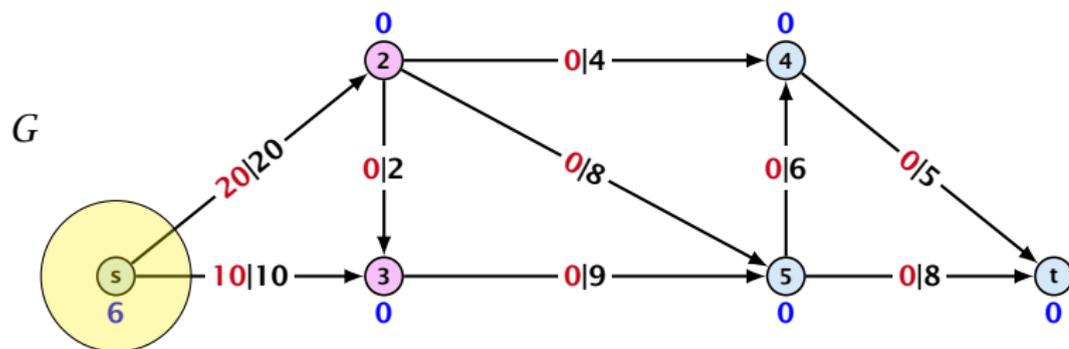
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- ▶ We have $s \in A$ and $t \in B$ and there is no edge from A to B in the residual graph G_f; this means that (A, B) is a saturated cut.

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Lemma 29

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

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Idea:

▶ start with some preflow and some valid labelling

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▶ successively change the preflow while maintaining a valid labelling

Push Relabel Algorithms

Idea:

- ▶ start with some preflow and some valid labelling
- ▶ successively change the preflow while maintaining a valid labelling
- ▶ stop when you have a flow (i.e., no more active nodes)

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissable if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation
Consider an active node u with excess flow $f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissable arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

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   the arc e is deleted from the residual graph

▶ non-saturating push: $\min\{f(u), c_f(e)\} = f(u)$
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Push Relabel Algorithms

The relabel operation
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The relabel operation
Consider an active node u that does not have an outgoing admissable arc.

Increasing the label of u by 1 results in a valid labelling.

- Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.

- An outgoing edge (u, w) had $\ell(u) < \ell(w) + 1$ before since it was not admissable. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:

We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water than this would be natural).

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- In a preflow nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- Such a node is called active.
- A labelling is valid if for every edge (u, v) in the residual graph $\ell(u) \leq \ell(v) + 1$.
- An arc (u, v) in residual graph is admissable if $\ell(u) = \ell(v) + 1$.
- A saturation push along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it dissappear from the residual graph).
- A non-saturating push along $e = (u, v)$ pushes a flow of $f(u)$, where $f(u)$ is the excess flow of u. This makes u inactive.

Push Relabel Algorithms

Algorithm 3 maxflow(G, s, t, c)

1: find initial preflow f

2: **while** there is active node u **do**

3:      **if** there is admiss. arc e out of u **then**

4:          push(G, e, f, c)

5:      **else**

6:          relabel(u)

7: **return** f

Push Relabel Algorithms

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In the following example we always stick to the same active node u until it becomes inactive but this is not required.
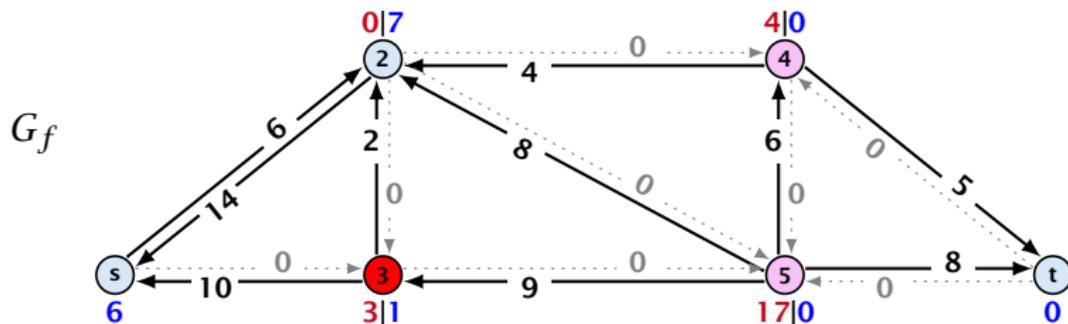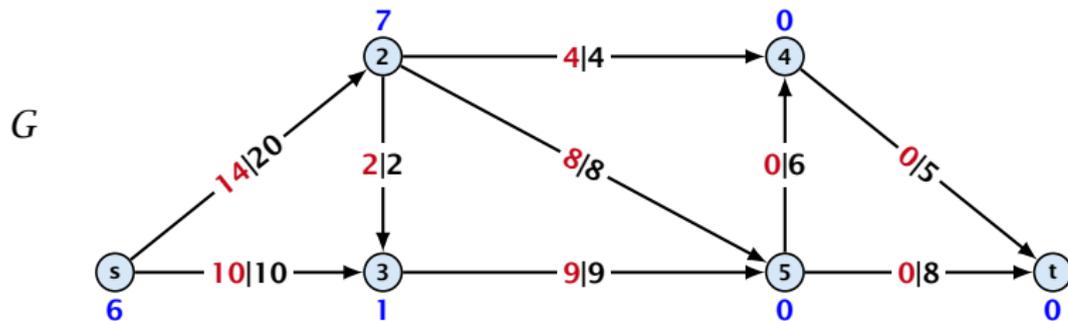
Preflow Push Algorithm

Preflow Push Algorithm

Preflow Push Algorithm

relabel

Preflow Push Algorithm

Preflow Push Algorithm

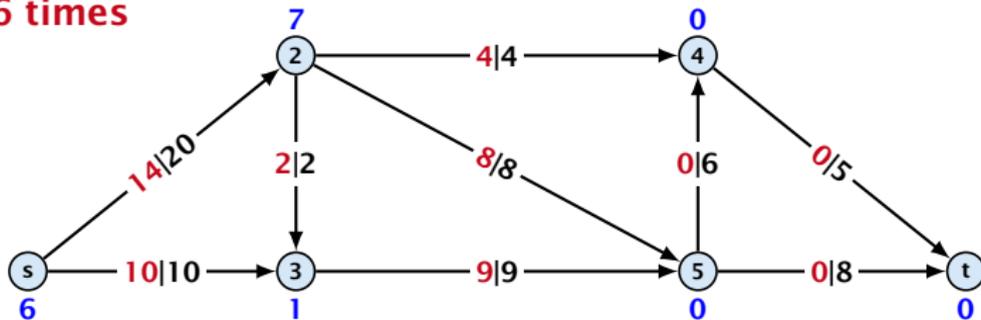**push**

Preflow Push Algorithm
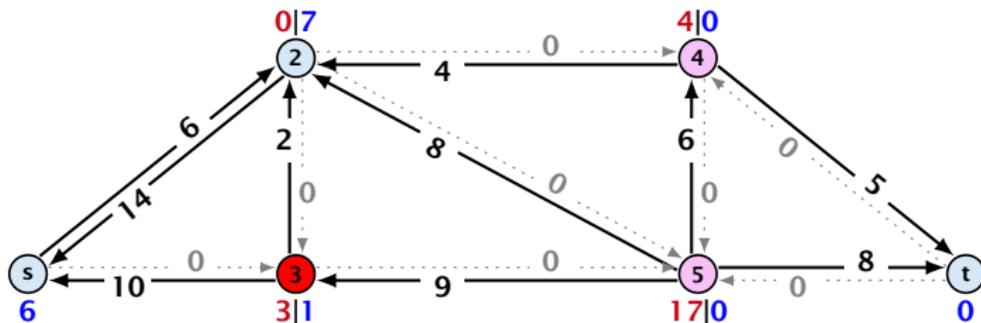
Preflow Push Algorithm

push

G

G_f

Preflow Push Algorithm

Preflow Push Algorithm

push

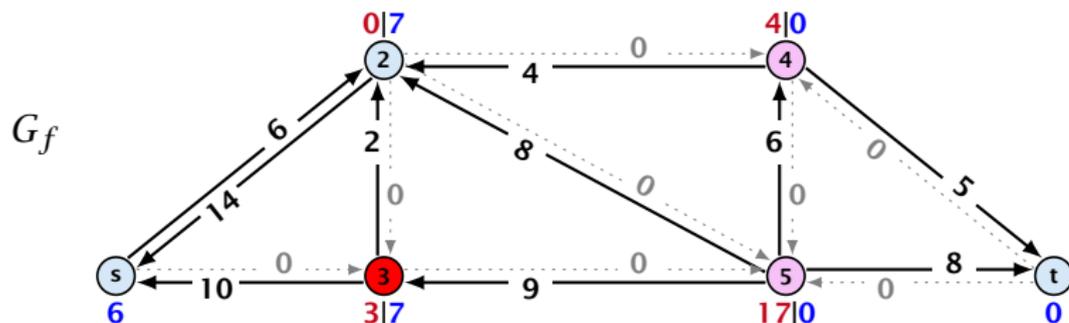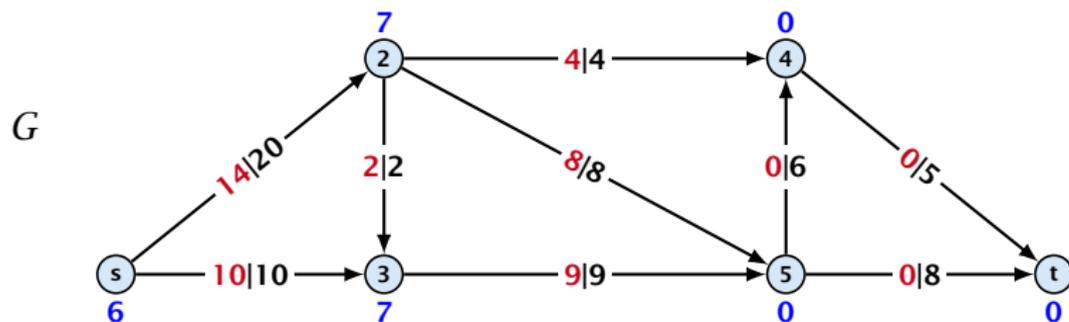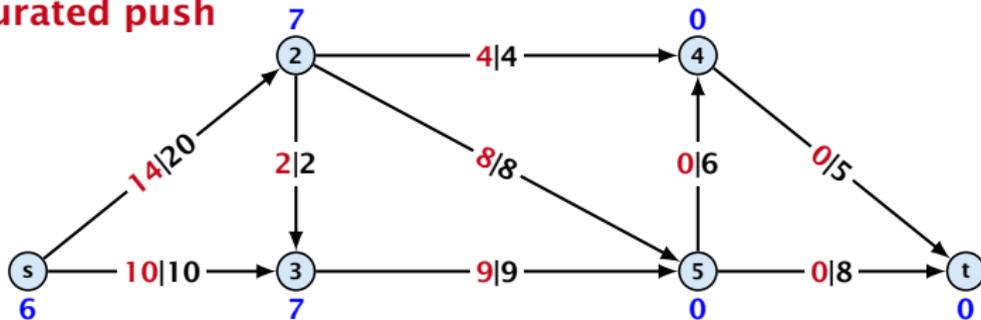# Preflow Push Algorithm

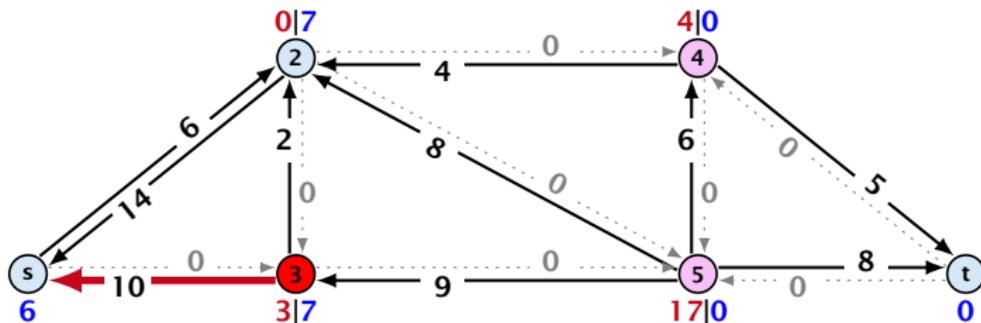# Preflow Push Algorithm

relabel 6 times

Preflow Push Algorithm

Preflow Push Algorithm

non-saturated push

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relabel

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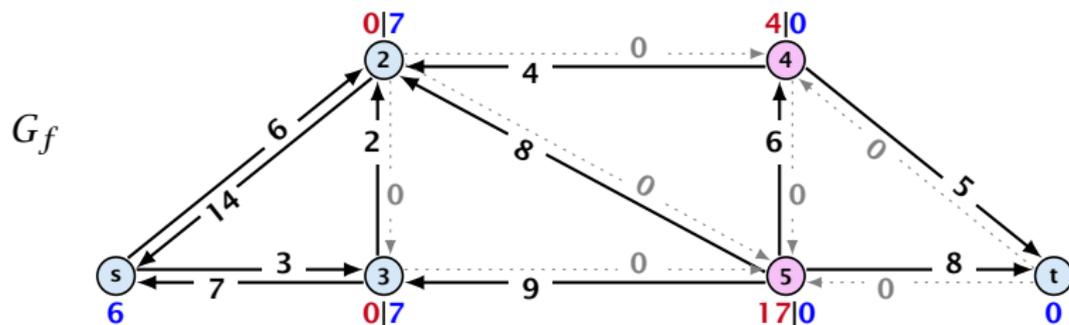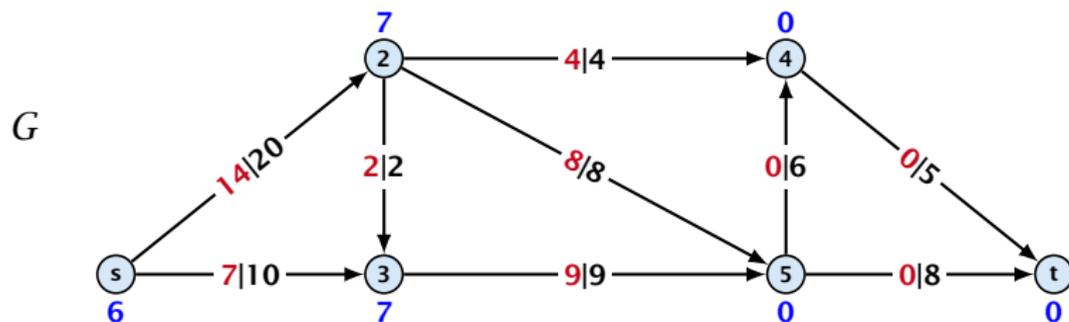# Preflow Push Algorithm

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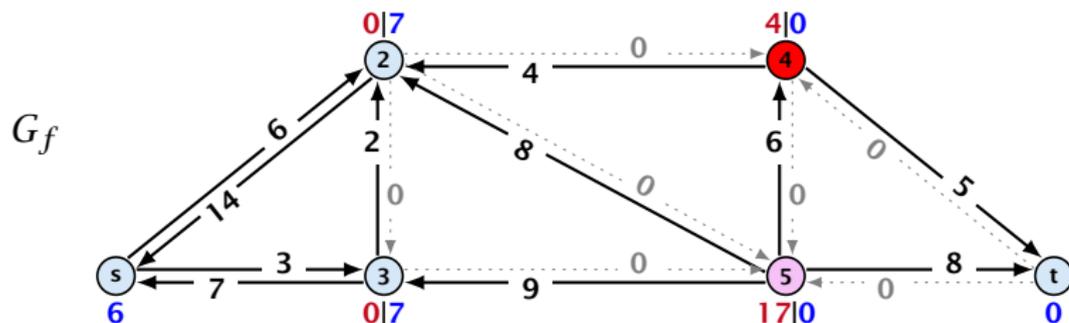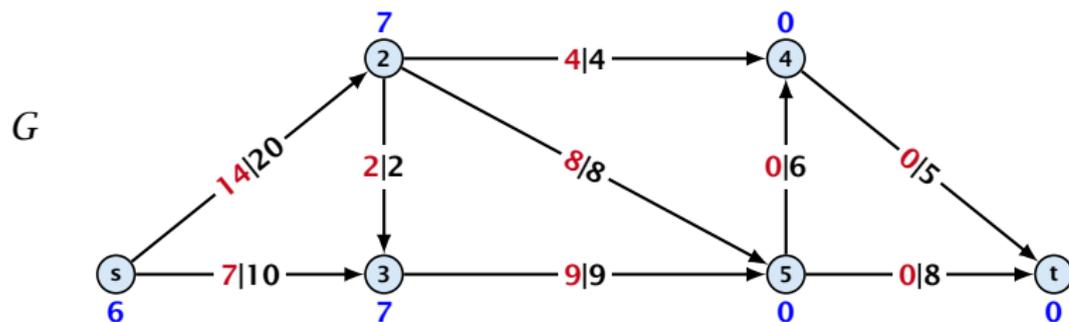
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G

G_f

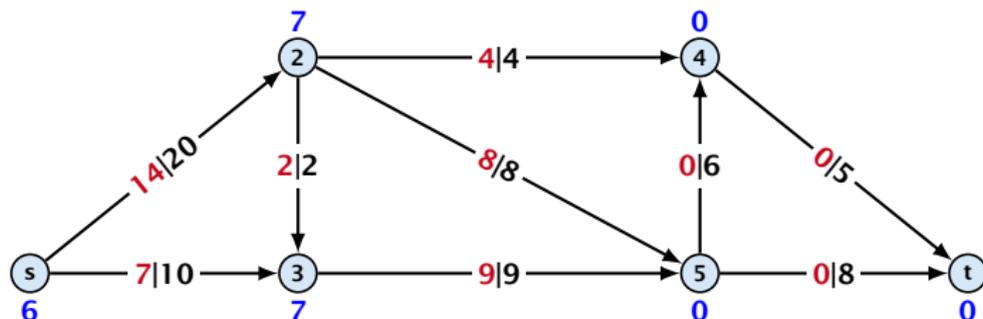# Preflow Push Algorithm

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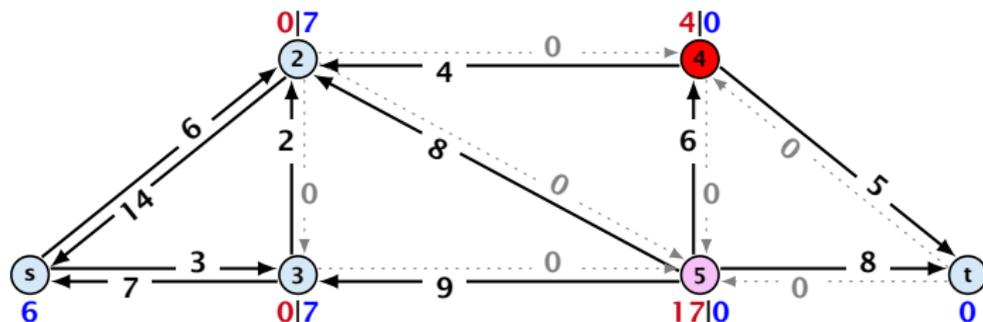## relabel 7 times

G
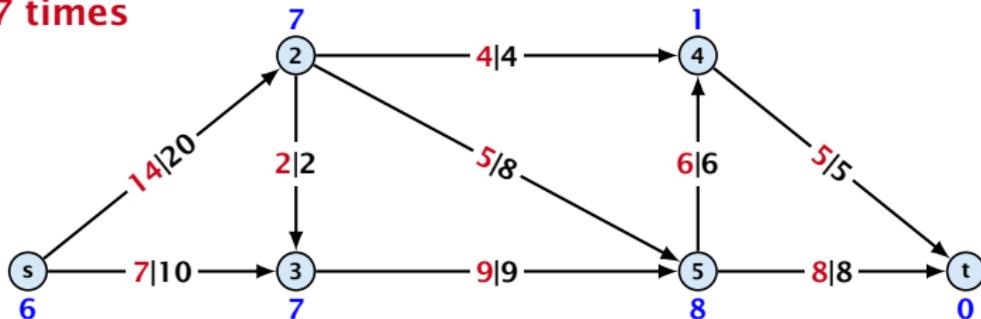
G_f
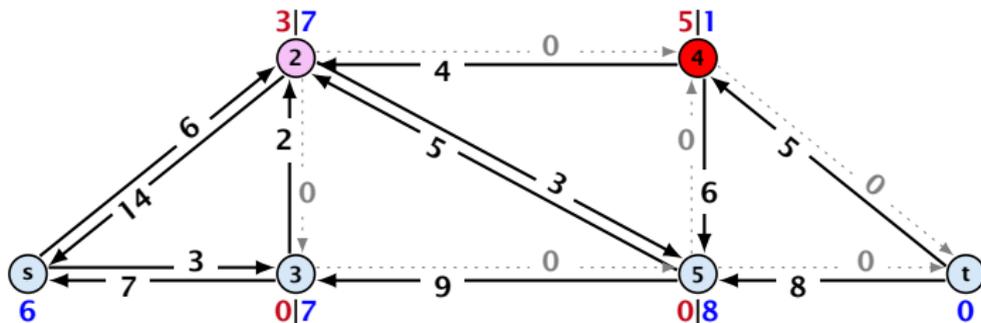
Preflow Push Algorithm
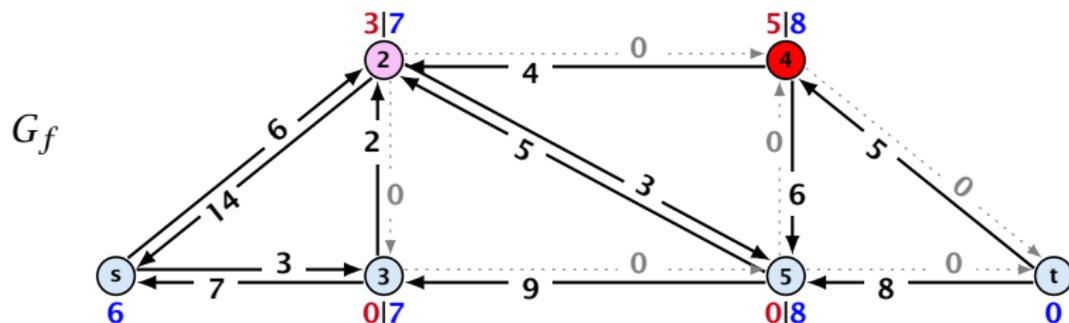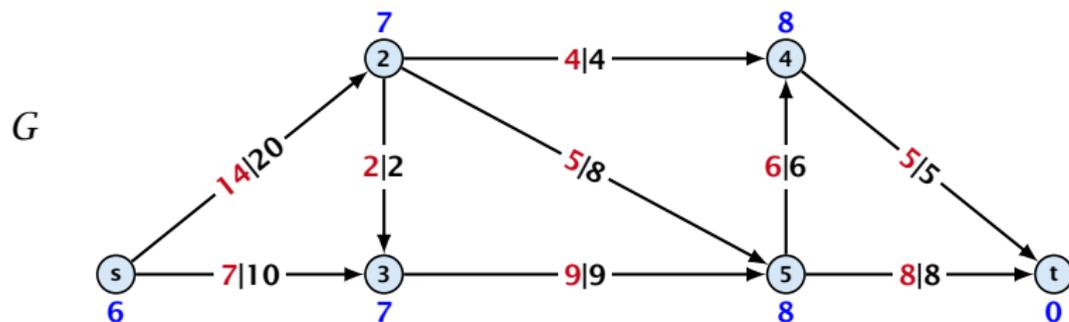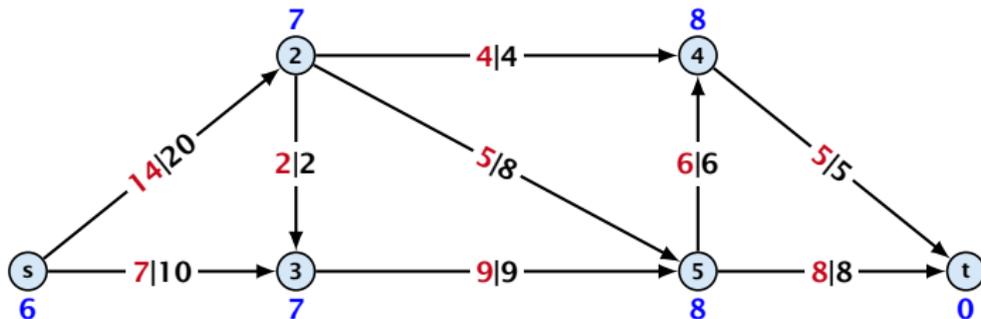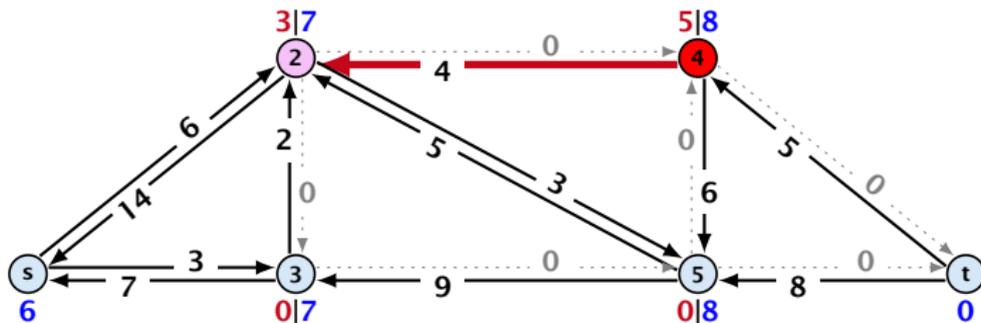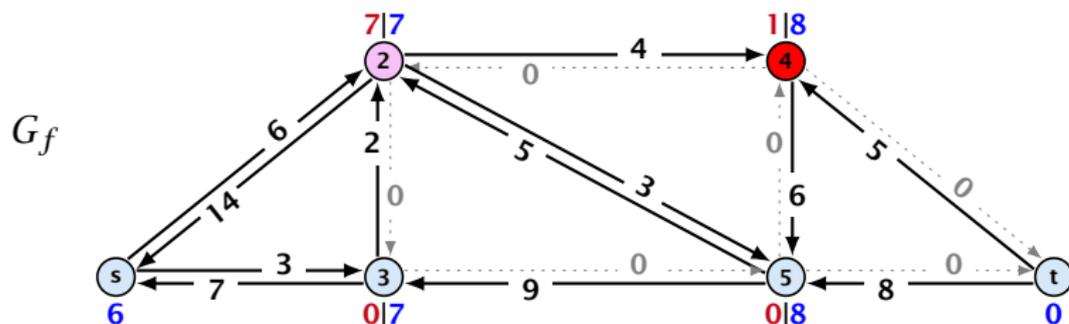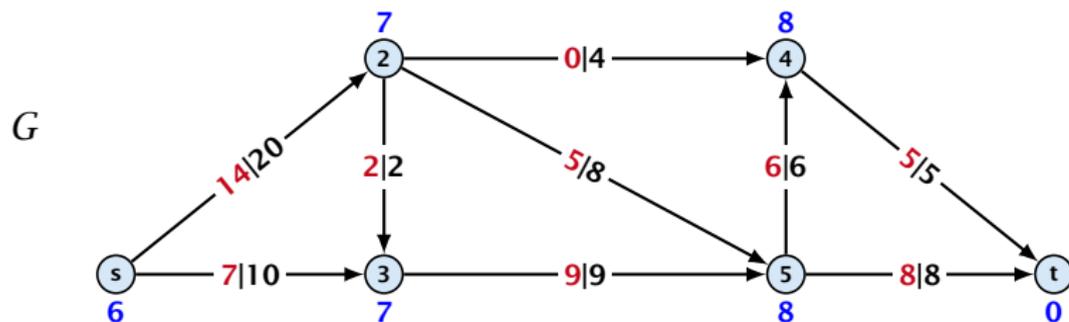
Preflow Push Algorithm

push

14.1 Generic Push Relabel
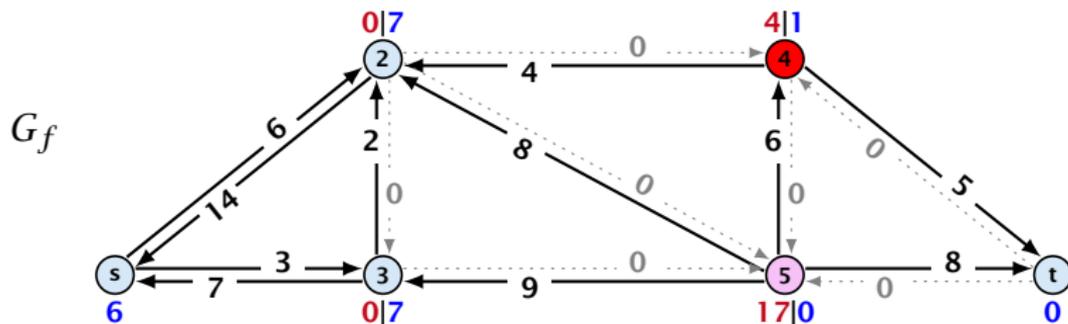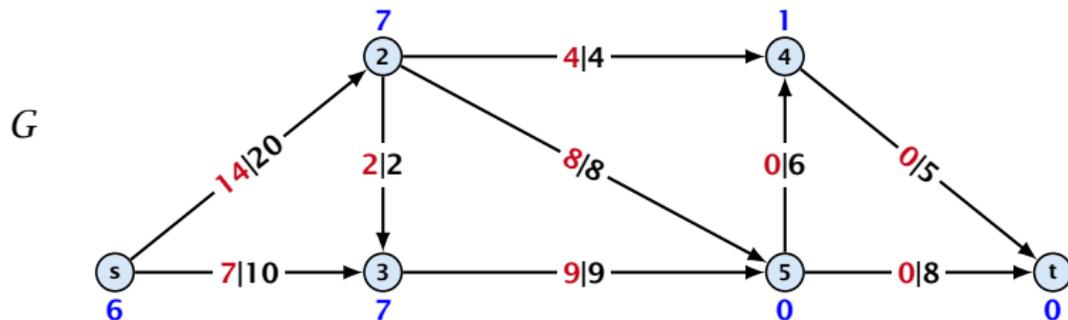
Preflow Push Algorithm

Preflow Push Algorithm

relabel
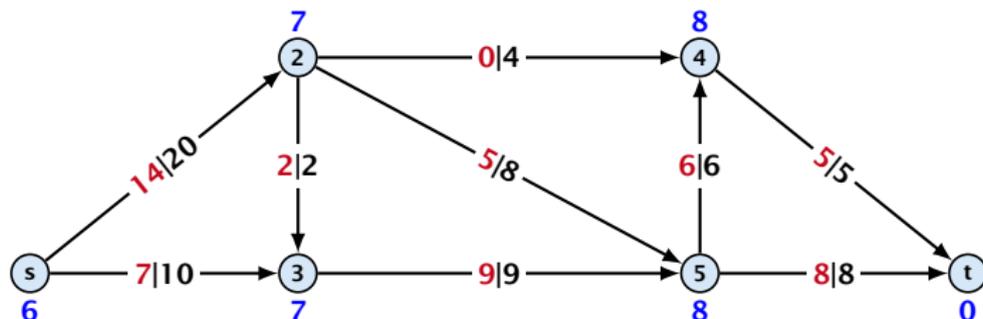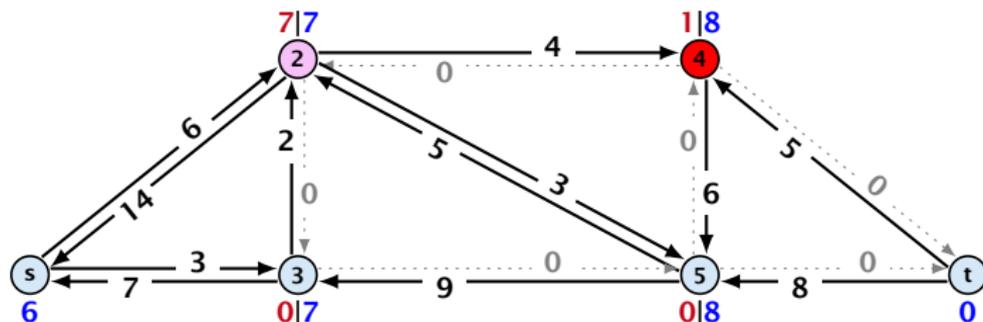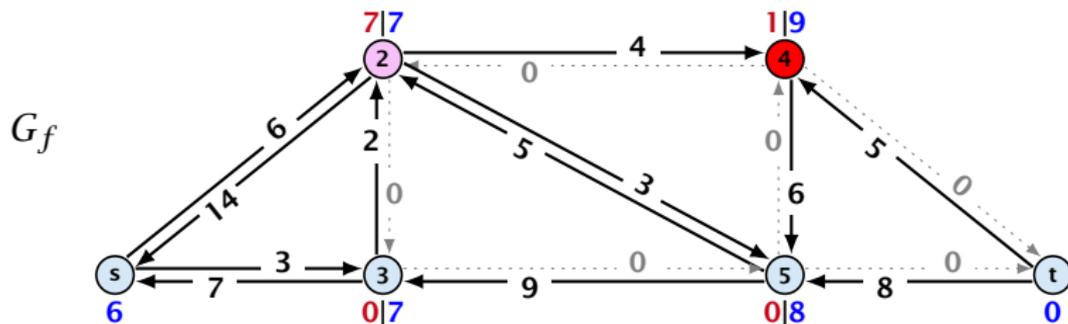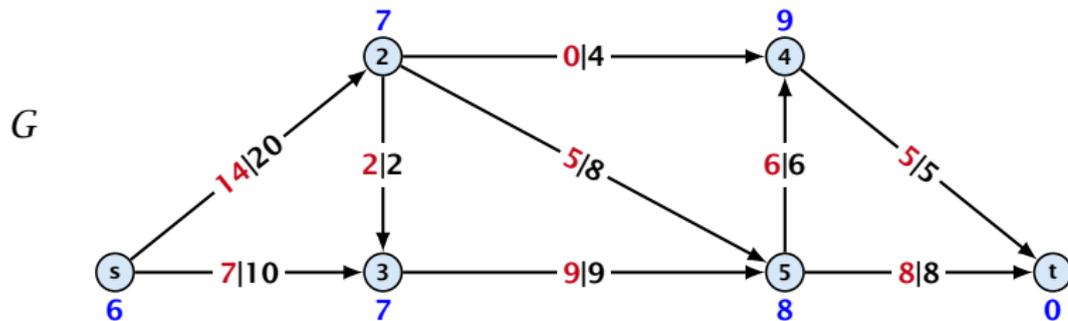
Preflow Push Algorithm

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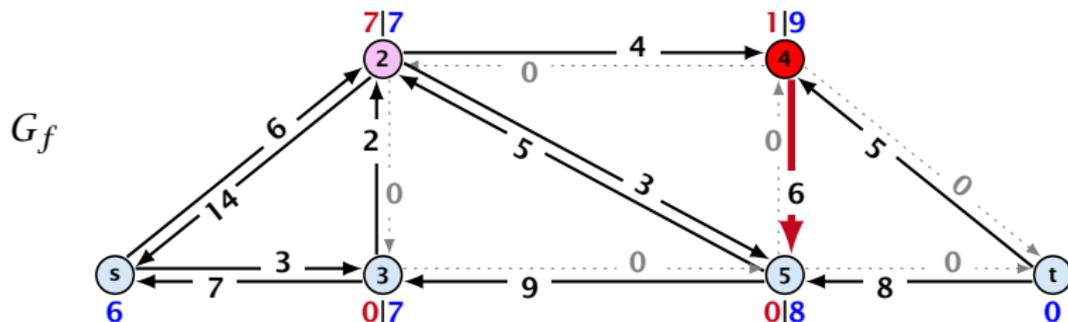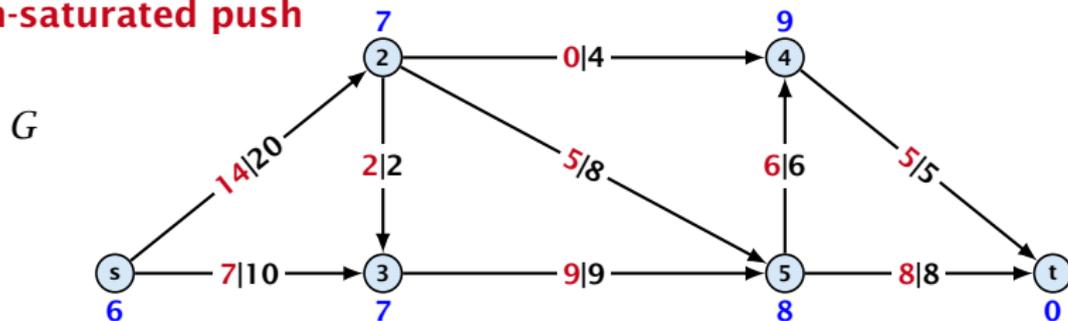
non-saturated push
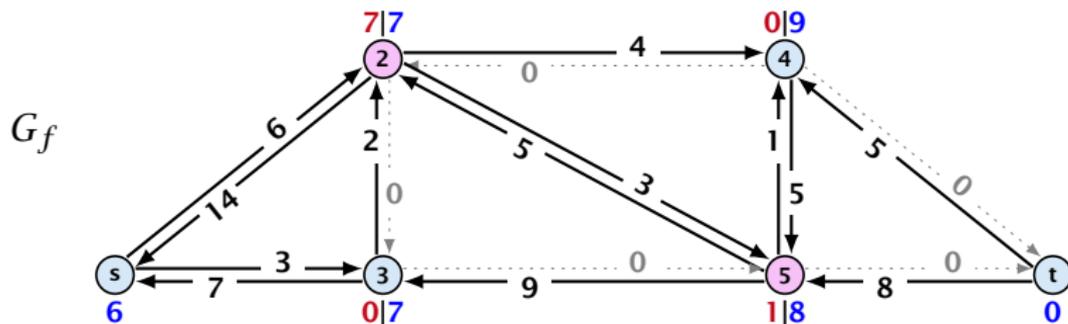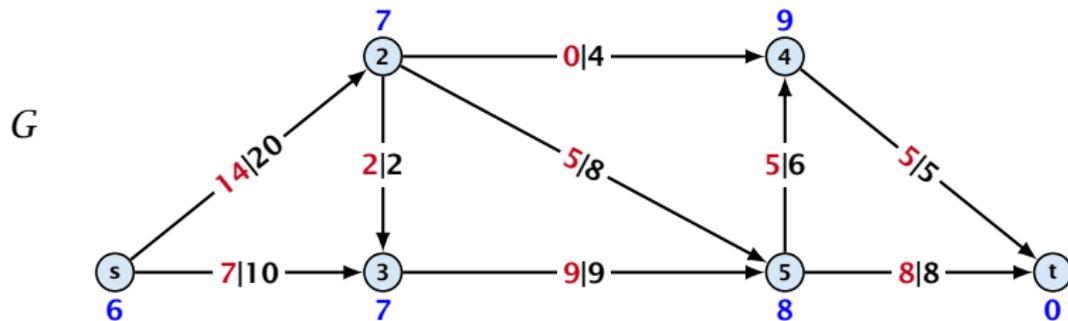
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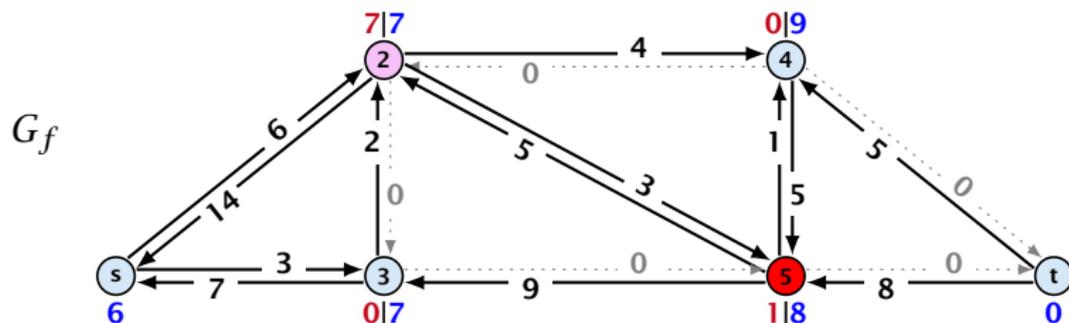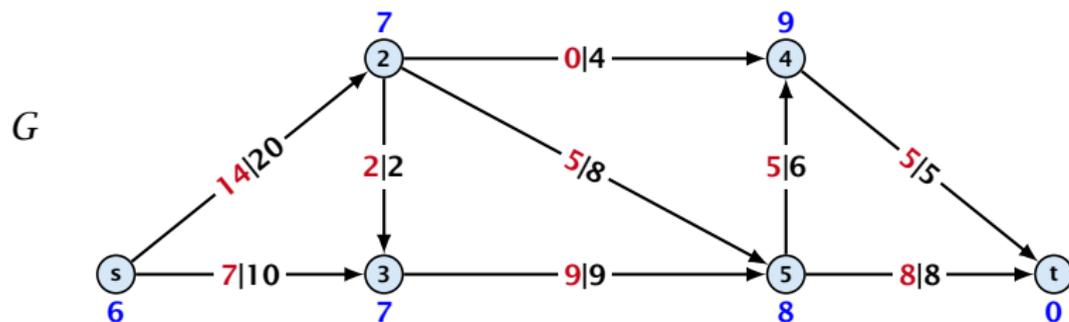
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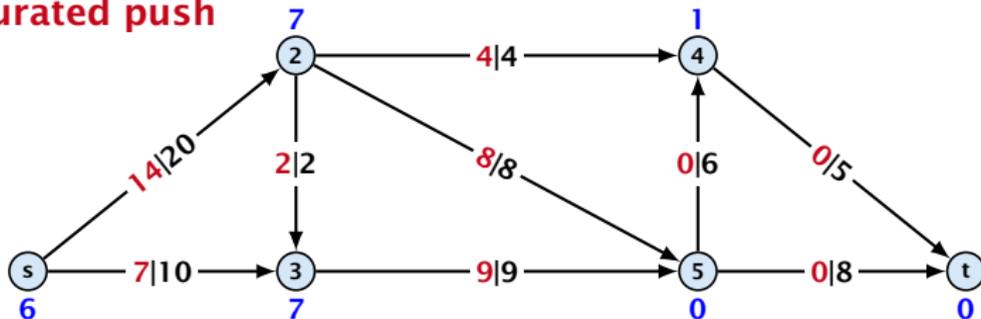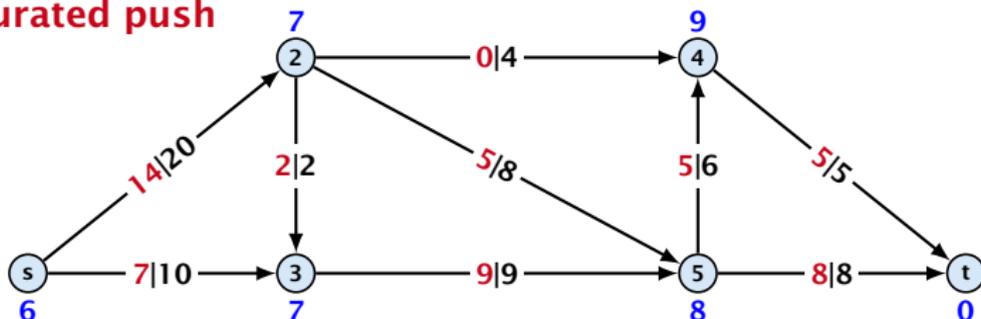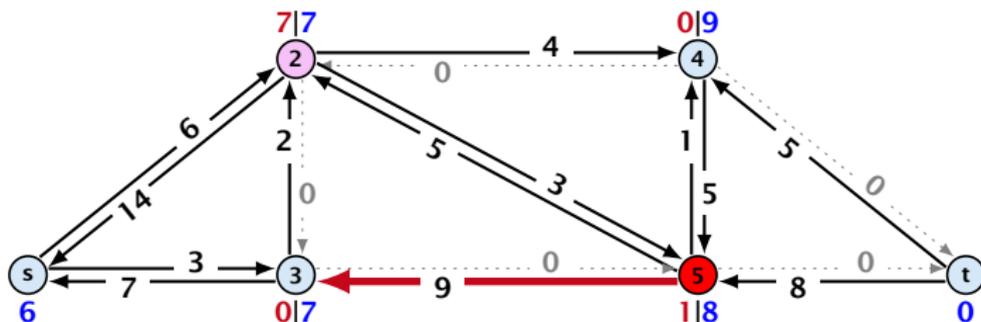
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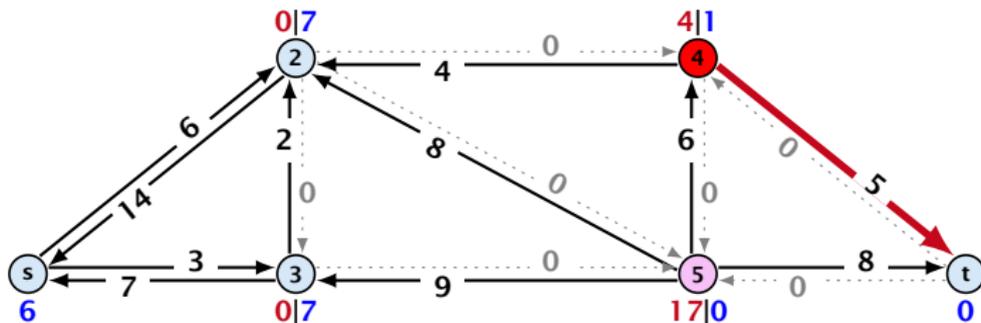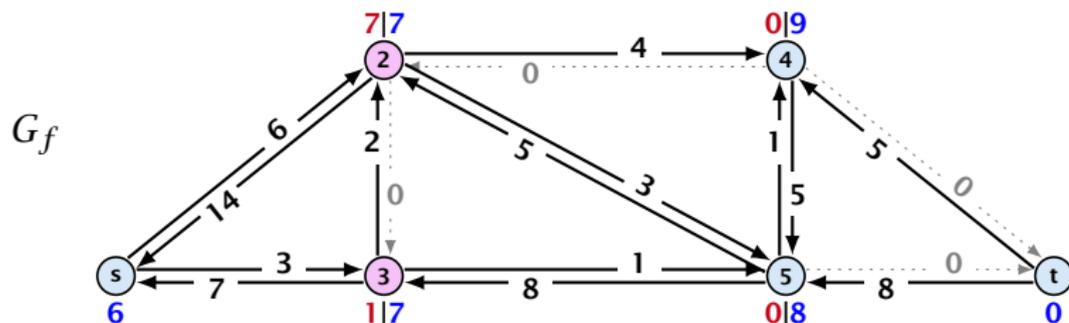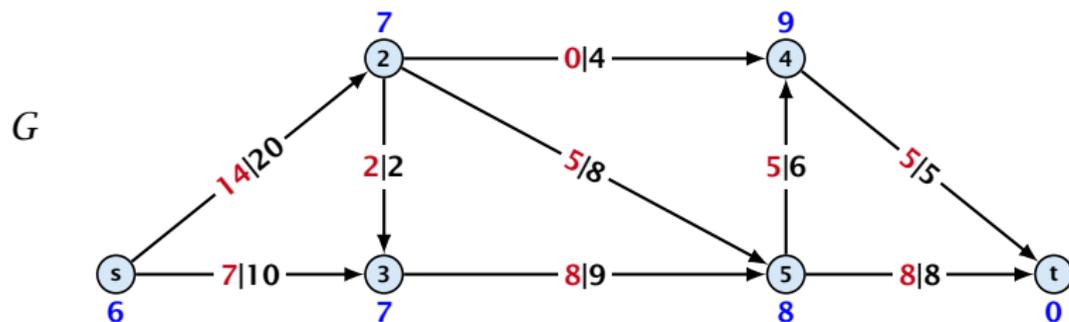
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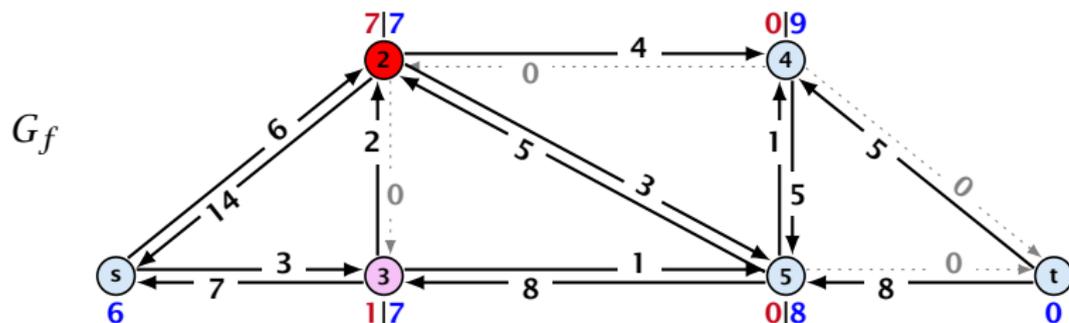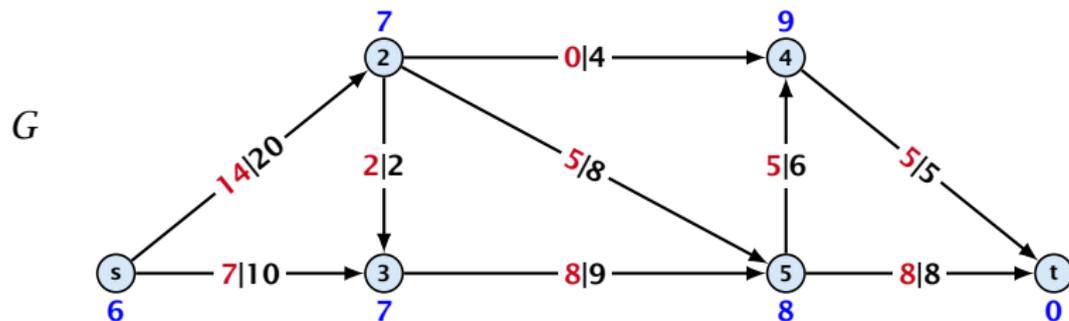
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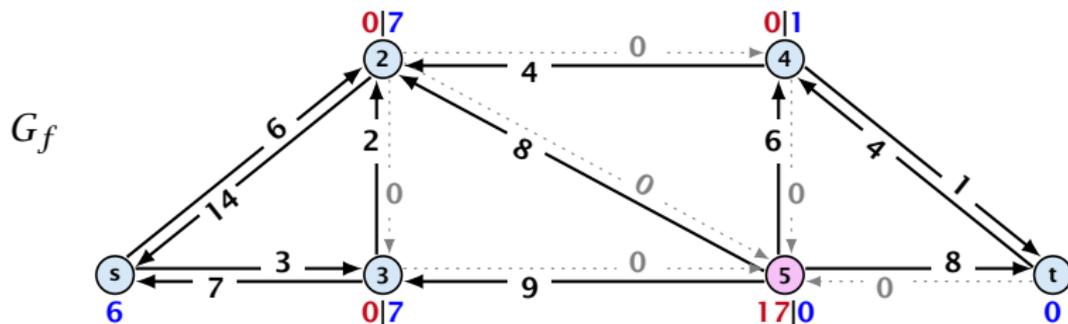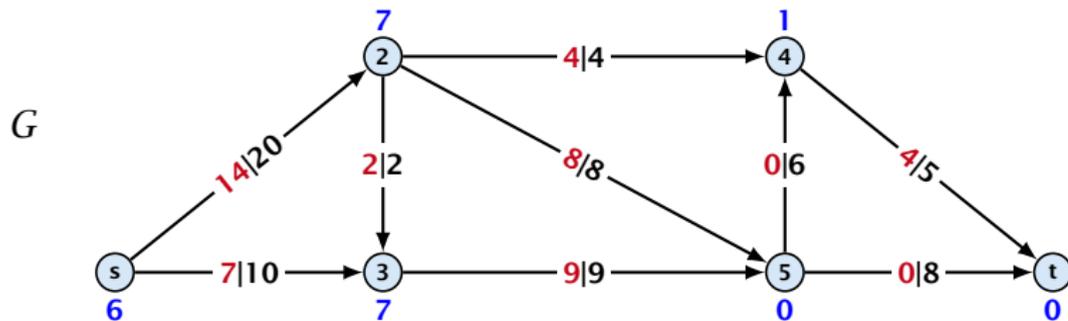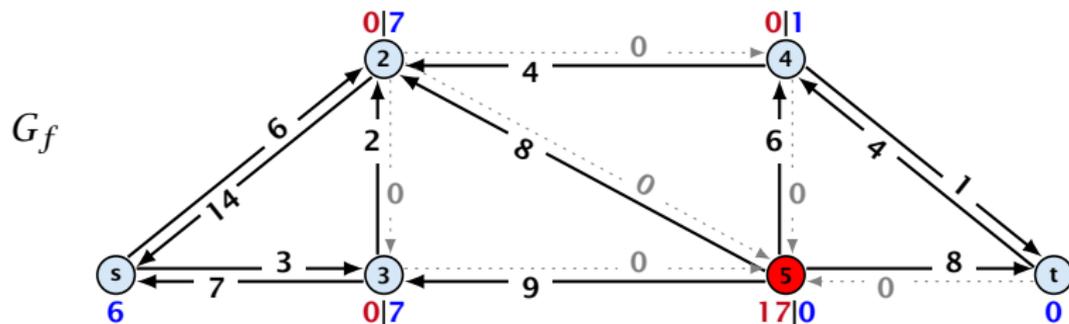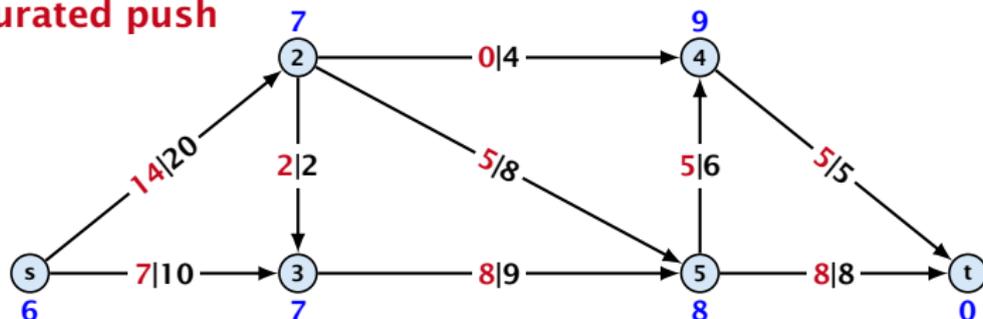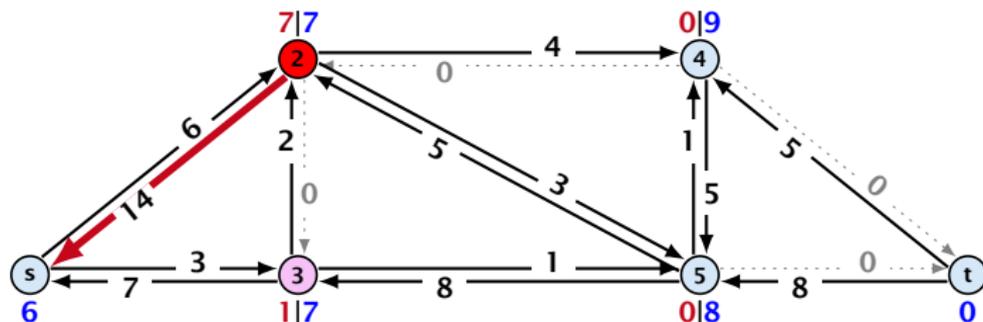
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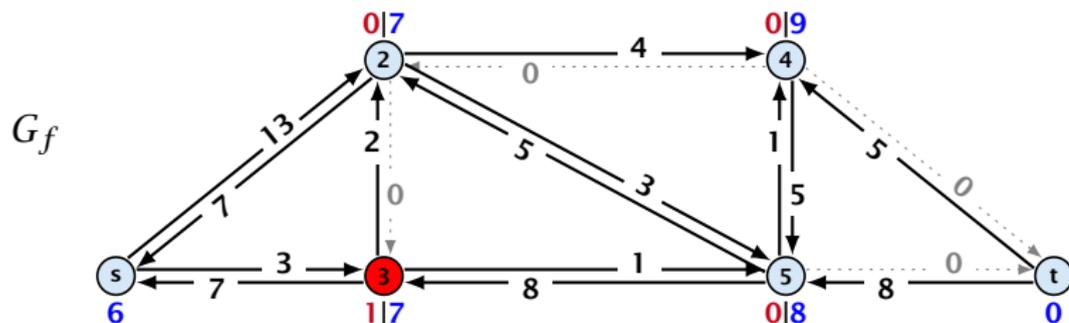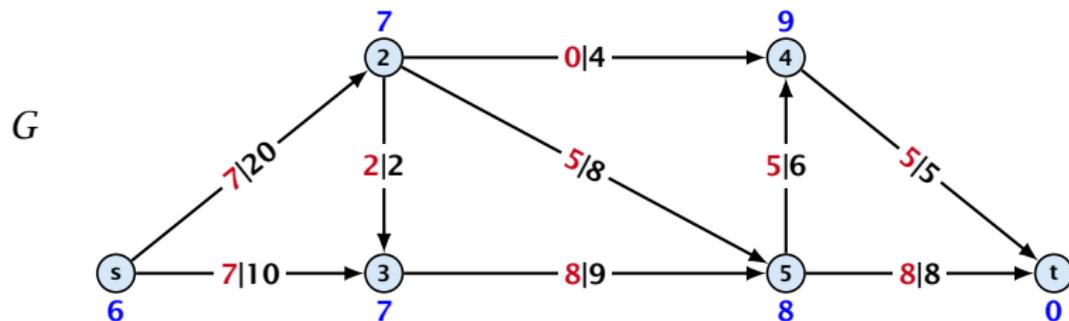
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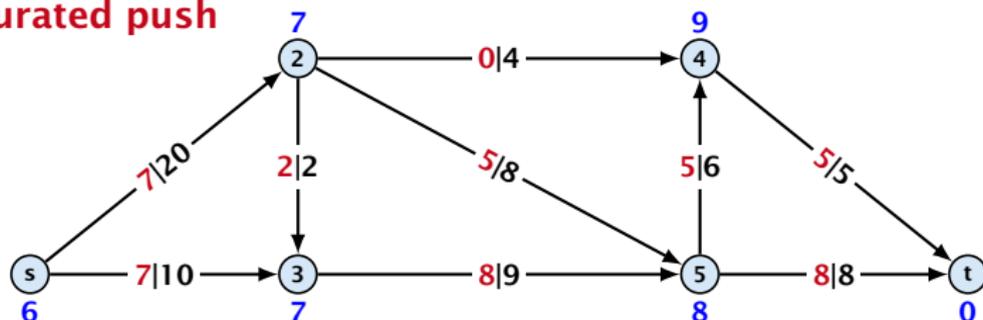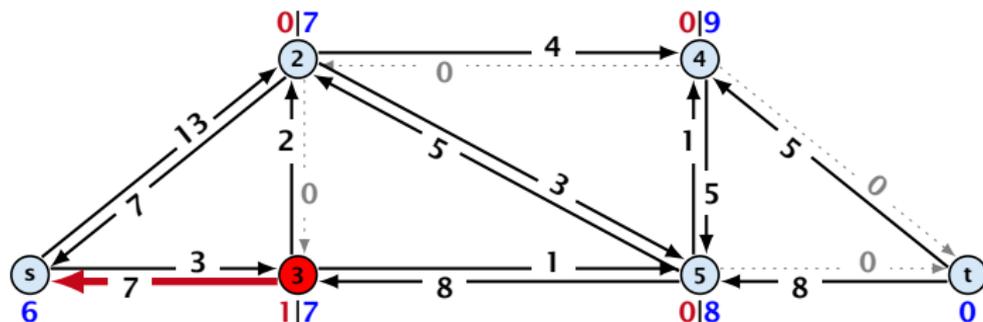
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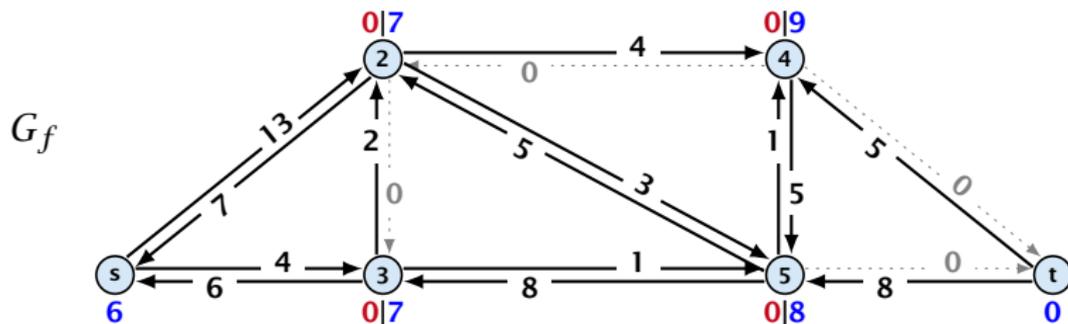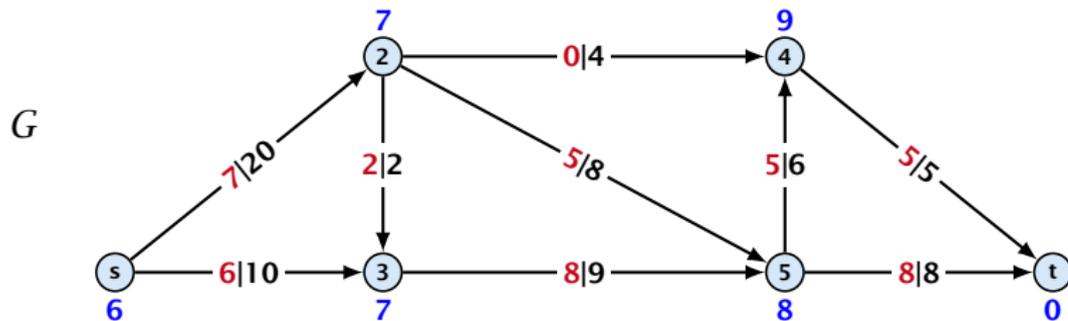
Analysis

Lemma 30

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- In the residual graph there are no edges into A, and, hence, no edges leaving A/entering B can carry any flow.
- Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

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$$
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&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right)
\end{aligned}
$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$
\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \sum_{b \in B} \sum_{v \in A} f(v, b) - \sum_{b \in B} \sum_{v \in A} f(b, v) + \sum_{b \in B} \sum_{v \in B} f(v, b) - \sum_{b \in B} \sum_{v \in B} f(b, v)
\end{aligned}
$$

14.1 Generic Push Relabel

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

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&= \sum_{b \in B} \sum_{v \in A} f(v, b) - \sum_{b \in B} \sum_{v \in A} f(b, v) + \boxed{\sum_{b \in B} \sum_{v \in B} f(v, b) - \sum_{b \in B} \sum_{v \in B} f(b, v)} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 0
\end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

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&= \sum_{b \in B} \sum_{v \in A} f(v, b) - \sum_{b \in B} \sum_{v \in A} f(b, v)
\end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$\begin{aligned} f(B) &= \sum_{b \in B} f(b) \\ &= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\ &= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\ &= \sum_{b \in B} \sum_{v \in A} \boxed{f(v, b)} - \sum_{b \in B} \sum_{v \in A} f(b, v) \\ & \qquad\qquad\quad \mathbf{= 0} \end{aligned}$$

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&= \boxed{\sum_{b \in B} \sum_{v \in A} f(v, b)} - \sum_{b \in B} \sum_{v \in A} f(b, v) \\
&\quad = 0
\end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

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$$f(B) = \sum_{b \in B} f(b)$$

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&= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} \boxed{f(b, v)} \\
&\qquad\qquad\qquad\qquad\quad \geq 0
\end{aligned}
$$

14.1 Generic Push Relabel

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

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We have

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\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
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&= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} f(b, v) \\
&\leq 0
\end{aligned}
$$

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&= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} f(b, v) \\
&\leq 0
\end{aligned}
$$

Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

14.1 Generic Push Relabel

Analysis

Lemma 31

The label of a node cannot become larger than $2n - 1$.

Analysis

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Proof.

▶ When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

Analysis

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from u to s of length at most $n - 1$. Along each edge of the
path the height/label can at most drop by 1, and the label
of the source is n.

Lemma 32

There are only $\mathcal{O}(n^2)$ relabel operations.

Analysis

Lemma 33

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Analysis

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Proof.

▶ Suppose that we just made a saturating push along (u, v).

Analysis

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Proof.

- ▶ Suppose that we just made a saturating push along (u, v).
- ▶ Hence, the edge (u, v) is deleted from the residual graph.

Analysis

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The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

- ▶ Suppose that we just made a saturating push along (u, v).
- ▶ Hence, the edge (u, v) is deleted from the residual graph.
- ▶ For the edge to appear again, a push from v to u is required.

Analysis

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The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

▶ Suppose that we just made a saturating push along (u, v).

▶ Hence, the edge (u, v) is deleted from the residual graph.

▶ For the edge to appear again, a push from v to u is required.

▶ Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissable edges.

Analysis

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The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

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▶ For a push from v to u the edge (v, u) must become admissable. The label of v must increase by at least 2.

Analysis

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▶ Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

Lemma 34

The number of non-saturating pushes performed is at most $\mathcal{O}(n^2 m)$.

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▸ Define a potential function $\Phi(f) = \sum_{\text{active nodes}v} \ell(v)$

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- Define a potential function $\Phi(f) = \sum_{\text{active nodes} v} \ell(v)$
- A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).

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▶ A non-saturating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.

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▶ Define a potential function $\Phi(f) = \sum_{\text{active nodes} v} \ell(v)$

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▶ A relabel increases Φ by at most 1.

▶ A non-saturating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.

▶ Hence,

$$\text{\#non-saturating\_pushes} \leq \text{\#relabels} + 2n \cdot \text{\#saturating\_pushes}$$
$$\leq \mathcal{O}(n^2 m) \ .$$

Analysis

Theorem 35

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2 m)$.

Analysis

Proof:

For every node maintain a list of admissable edges starting at
that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

A relabel at a node u can be performed in time $\mathcal{O}(n)$

Analysis

Proof:

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

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A relabel at a node u can be performed in time $\mathcal{O}(n)$

Analysis

Proof:

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
- ▶ check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- ▶ check list of outgoing edges of u for admissible edges after
- ▶ the list of incoming edges of u and update the corresponding edges

Analysis

Proof:

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- check whether edge (v, u) needs to be added to G_f
- check whether (u, v) needs to be deleted (saturating push)
- check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$
- check first of outgoing edges in the alternative admissable edge
- give a set of incoming edges which becomes the outgoing edges

Analysis

Proof:

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
- ▶ check whether u becomes inactive and has to be deleted from the set of active nodes

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Analysis

Proof:

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
- ▶ check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissable
- ▶ check for all incoming edges if they become non-admissable

Analysis

Proof:

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
- ▶ check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissable
- ▶ check for all incoming edges if they become non-admissable

Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 4 discharge(u)

1: **while** u is active **do**
2:     $v \leftarrow u.current\text{-}neighbour$
3:     **if** $v = \text{null}$ **then**
4:         relabel(u)
5:         $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
6:     **else**
7:         **if** (u, v) admissable **then** push(u, v)
8:         **else** $u.current\text{-}neighbour \leftarrow v.next\text{-}in\text{-}list$

Note that $u.current\text{-}neighbour$ is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 36
If $v = \text{null}$ in Line 3, then there is no outgoing admissable edge from u.

Proof.

▶ While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissable.

▶ The only thing that could make the edge admissable again would be a relabel at u.

▶ If we reach the end of the list ($v = \text{null}$) all edges are not admissable. □

This shows that discharge(u) is correct, and that we can perform a relabel in Line 4.

14.2 Relabel to Front

Algorithm 50 relabel-to-front(G, s, t)

1: initialize preflow
2: initialize node list L containing $V \setminus \{s, t\}$ in any order
3: **foreach** $u \in V \setminus \{s, t\}$ **do**
4:      $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
5: $u \leftarrow L.head$
6: **while** $u \neq$ null **do**
7:      $old\text{-}height \leftarrow \ell(u)$
8:      discharge(u)
9:      **if** $\ell(u) > old\text{-}height$ **then** // relabel happened
10:          move u to the front of L
11:      $u \leftarrow u.next$

14.2 Relabel to Front

Lemma 37 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissable edges; this means for an admissable edge (x, y) the node x appears before y in sequence L.*

2. *No node before u in the list L is active.*

Proof:

- ▶ Initialization:
    1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissable, which means that any ordering L is permitted.
    2. We start with u being the head of the list; hence no node before u can be active

- ▶ Maintenance:
    1. ▶ Pushes do no create any new admissable edges. Therefore, if discharge() does not relabel u, L is still topologically sorted.
       ▶ After relabeling, u cannot have admissable incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissable edges leaving u that were generated by the relabeling.

14.2 Relabel to Front

Proof:

▶ Maintenance:

    2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

    For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissable arc. However, all admissable arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

14.2 Relabel to Front

Lemma 38

There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

14.2 Relabel to Front

Lemma 39

The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting *u.current-neighbour*). In total we have $\mathcal{O}(n^2)$ relabel-operations.

14.2 Relabel to Front

Note that by definition a saturing push operation
($\min\{c_f(e), f(u)\} = c_f(e)$) can at the same time be a
non-saturating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 40
*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but
makes the edge e disappear from the residual graph. Therefore
the push-operation is immediately followed by an increase of the
pointer $u.current\text{-}neighbour$.
This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times
(upper bound on number of relabels) and the neighbour-list has
only $degree(u) + 1$ many entries ($+1$ for null-entry).

14.2 Relabel to Front

Lemma 41

The cost for all non-saturating push-operations is only $\mathcal{O}(n^3)$.

A non-saturating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 42

The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

14.3 Highest label

Algorithm 6 highest-label(G, s, t)
1: initialize preflow
2: **foreach** $u \in V \setminus \{s, t\}$ **do**
3:      $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
4: **while** $\exists$ active node u **do**
5:      select active node u with highest label
6:      discharge(u)

14.3 Highest label

Lemma 43

When using highest label the number of non-saturating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only "activate" nodes on levels strictly less than ℓ.

This means, after a non-saturating push from u a relabel is required to make u active again.

Hence, after n non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

14.3 Highest label

Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:
How do we find the next node for a discharge operation?

14.3 Highest label

Maintain lists L_i, $i \in \{0, \ldots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \ldots, L_0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to s or t the list $k - 1$ must be non-empty (i.e., the search takes constant time).

14.3 Highest label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#\text{non-saturating-pushes-to-s-or-t})$$

Lemma 44

The number of non-saturating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 45

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

14.3 Highest label

Proof of the Lemma.

▶ We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.

▶ After a node v (which must have $\ell(v) = n + 1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.

▶ This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.

▶ As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

Mincost Flow

Problem Definition:

$$
\begin{aligned}
\min \quad & \textstyle\sum_e c(e) f(e) \\
\text{s.t.} \quad & \forall e \in E : \ 0 \le f(e) \le u(e) \\
& \forall v \in V : \ f(v) = b(v)
\end{aligned}
$$

Mincost Flow

Problem Definition:

$$\min \quad \sum_e c(e)f(e)$$
$$\text{s.t.} \quad \forall e \in E: \ 0 \le f(e) \le u(e)$$
$$\forall v \in V: \ f(v) = b(v)$$

▶ $G = (V, E)$ is a directed graph.

▶ $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.

▶ $c : E \to \mathbb{R}$ is the cost function
  (note that $c(e)$ may be negative).

▶ $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Mincost Flow

Problem Definition:

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\min \quad & \sum_e c(e) f(e) \\
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- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Mincost Flow

Problem Definition:

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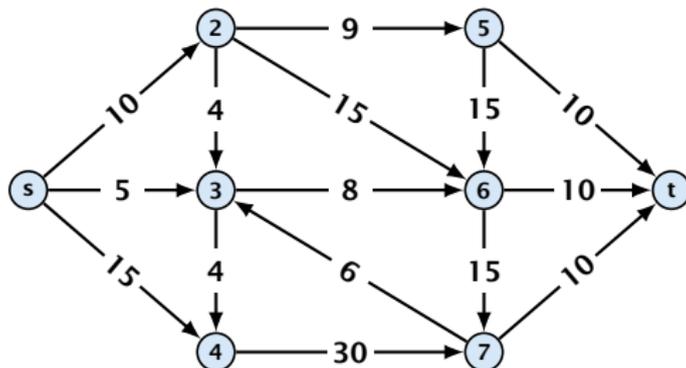
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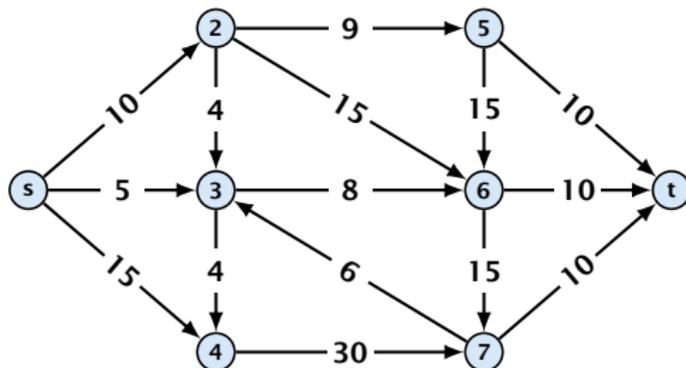
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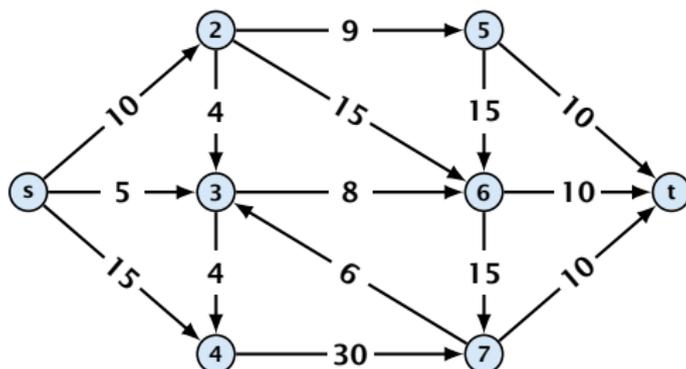
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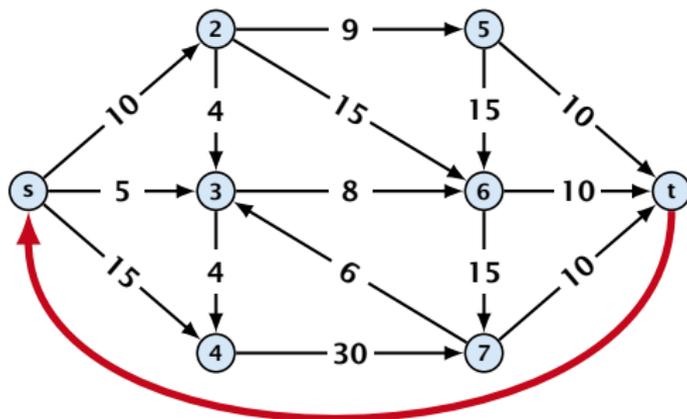
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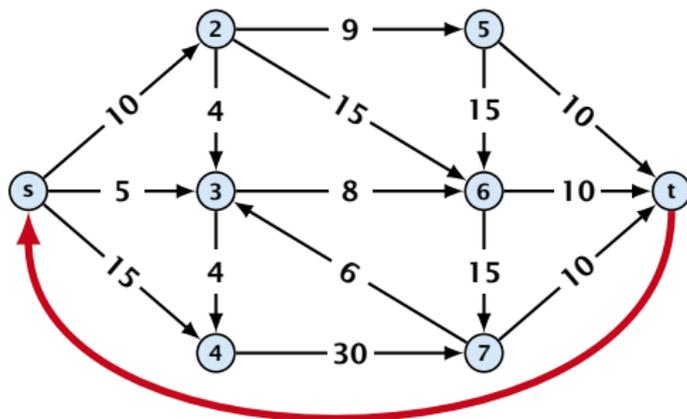
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- ▶ Then, $\mathrm{val}(f^*) = -\mathrm{cost}(f_{\min})$, where f^* is a maxflow, and $f_{\min}$ is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.

- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.

- Set edge-costs to zero, and keep the capacities.

- There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

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Generalization

Our model:

$$\begin{aligned} \min \quad & \sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E : \ 0 \le f(e) \le u(e) \\ & \forall v \in V : \ f(v) = b(v) \end{aligned}$$

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A more general model?

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Generalization

Differences

- ▶ Flow along an edge e may have non-zero lower bound $\ell(e)$.
- ▶ Flow along e may have negative upper bound $u(e)$.
- ▶ The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

min $\sum_e c(e) f(e)$

s.t. $\forall e \in E : \quad \ell(e) \leq f(e) \leq u(e)$

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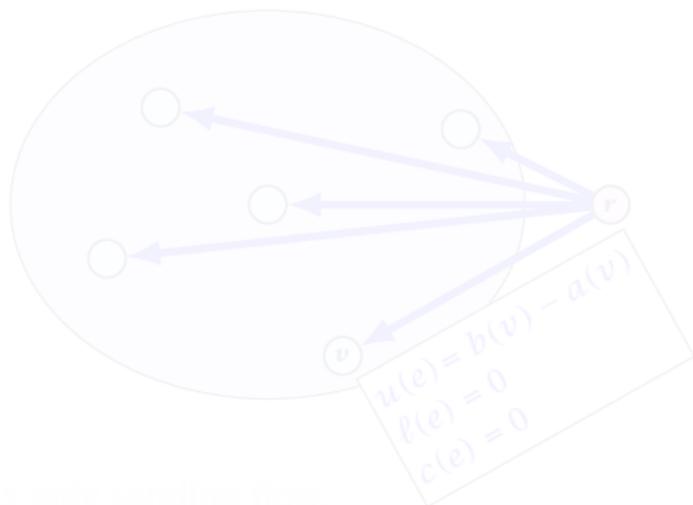
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Add edge (r, v) for all $v \in V$.

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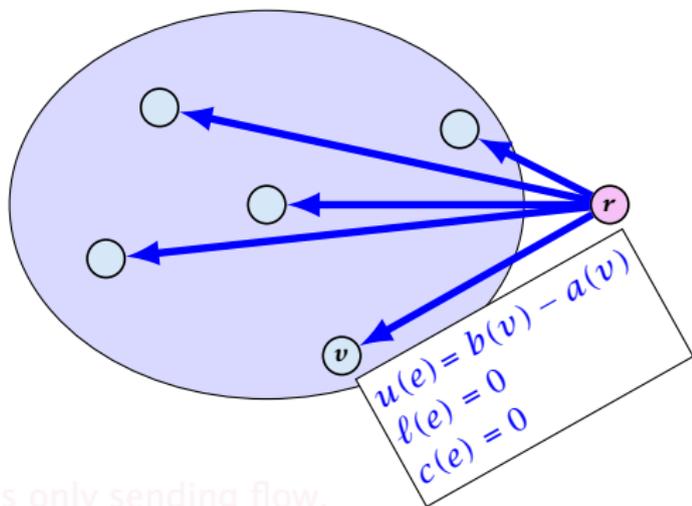Add edge (r, v) for all $v \in V$.

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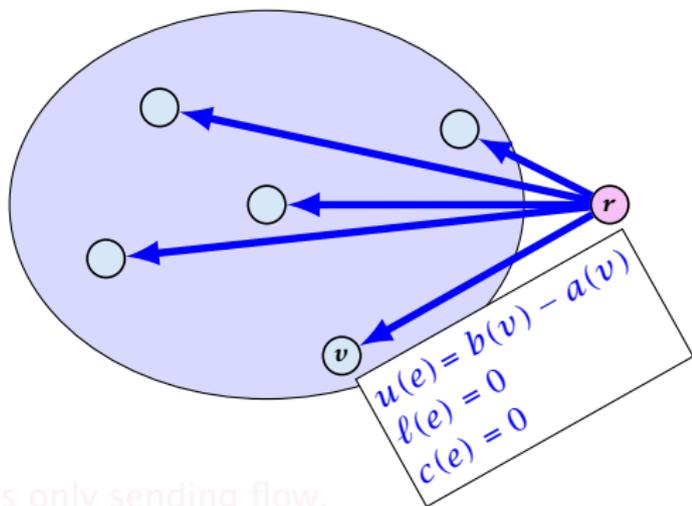
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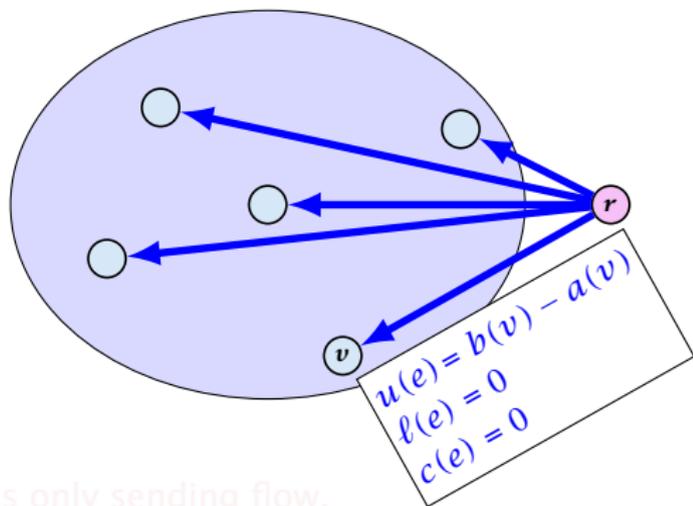Add edge (r, v) for all $v \in V$.

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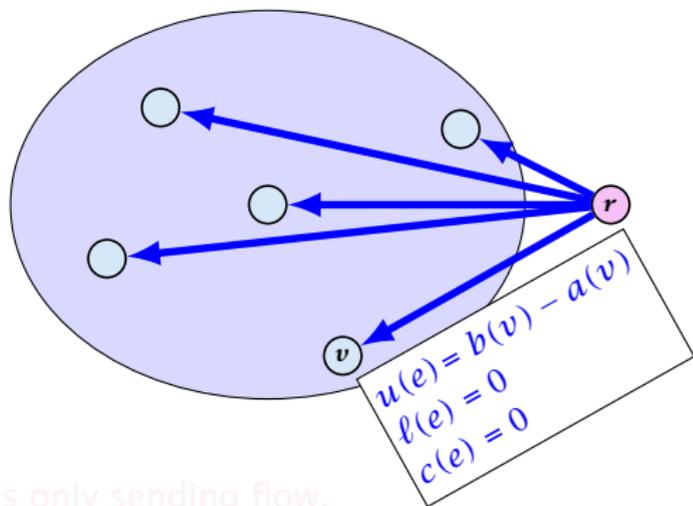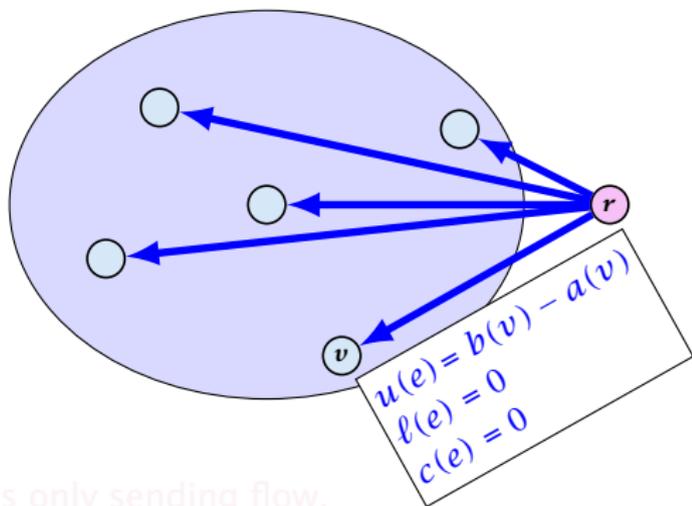Add edge (r, v) for all $v \in V$.

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We can assume that either $\ell(e) \ne -\infty$ or $u(e) \ne \infty$:

$$u(e) = \infty$$
$$\ell(e) = -\infty$$
$$c(e) = 0$$

If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \ne 0$ we can transform the graph so that $c(e) = 0$.

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We can transform any network so that a particular edge has cost $c(e) = 0$:

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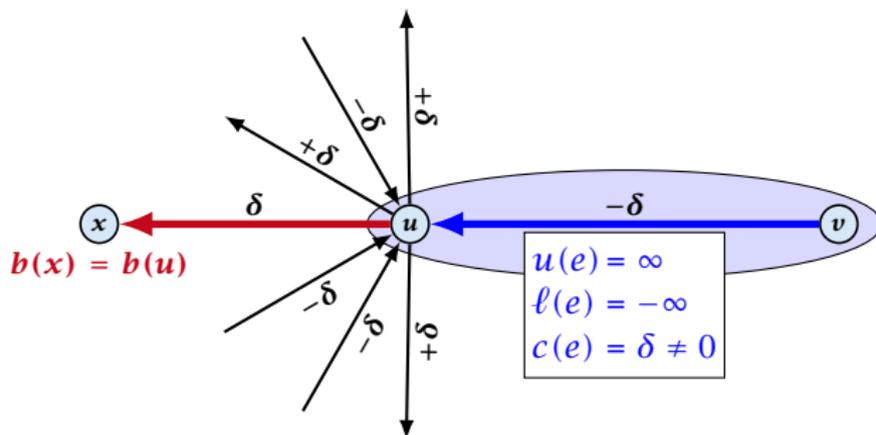
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Reduction III

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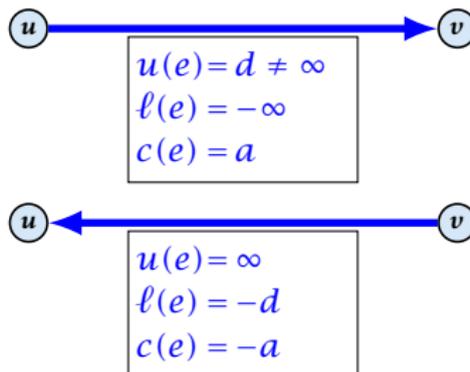
We can assume that $\ell(e) \ne -\infty$:

Replace the edge by an edge in opposite direction.

Reduction IV

$$\min \quad \sum_e c(e) f(e)$$
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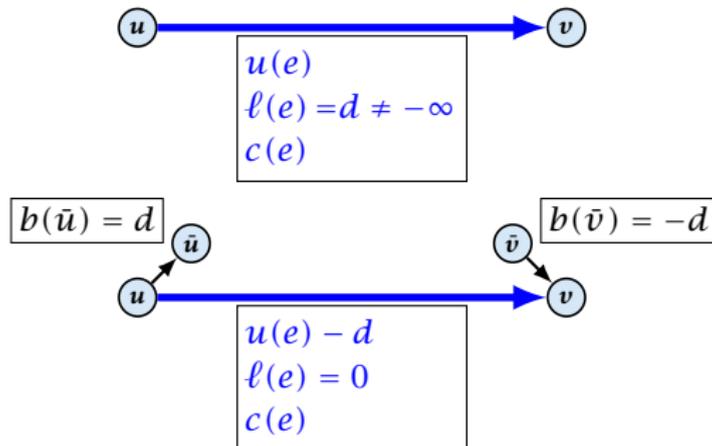
The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

▶ She needs to supply r_i napkins on N successive days.

▶ She can buy new napkins at p cents each.

▶ She can launder them at a fast laundry that takes m days and cost f cents a napkin.

▶ She can use a slow laundry that takes $k > m$ days and costs s cents each.

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15 Mincost Flow

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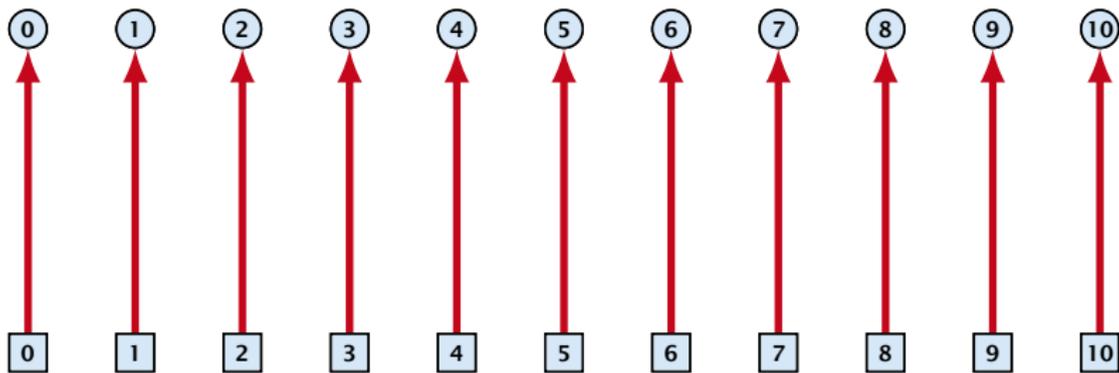
▶ Minimize cost.

day edges:

| upper bound: $u(e_i) = \infty$; |
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| cost: $c(e) = 0$ |
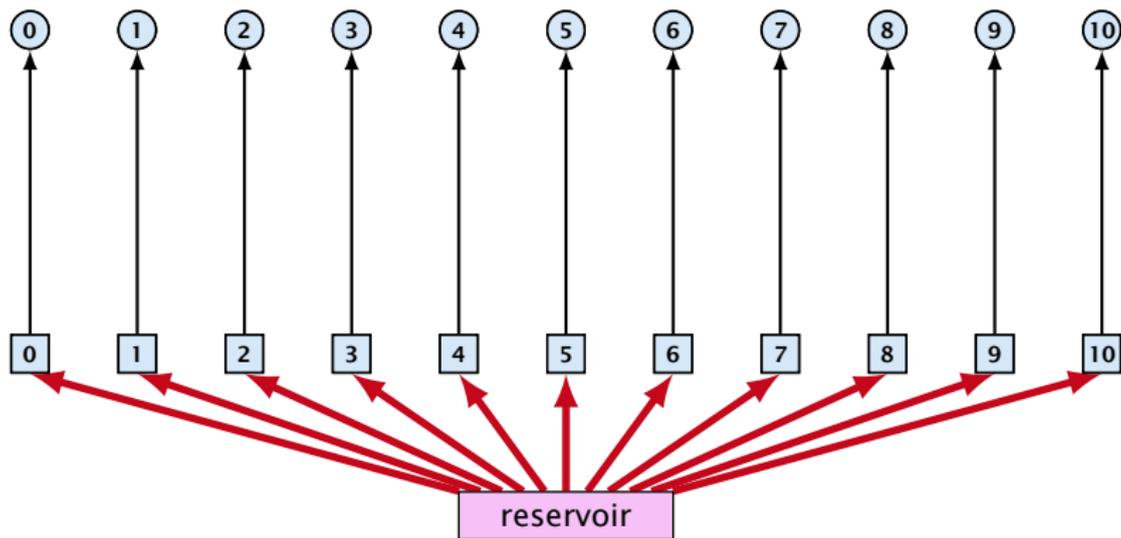
buy edges:

upper bound: $u(e_i) = \infty$;
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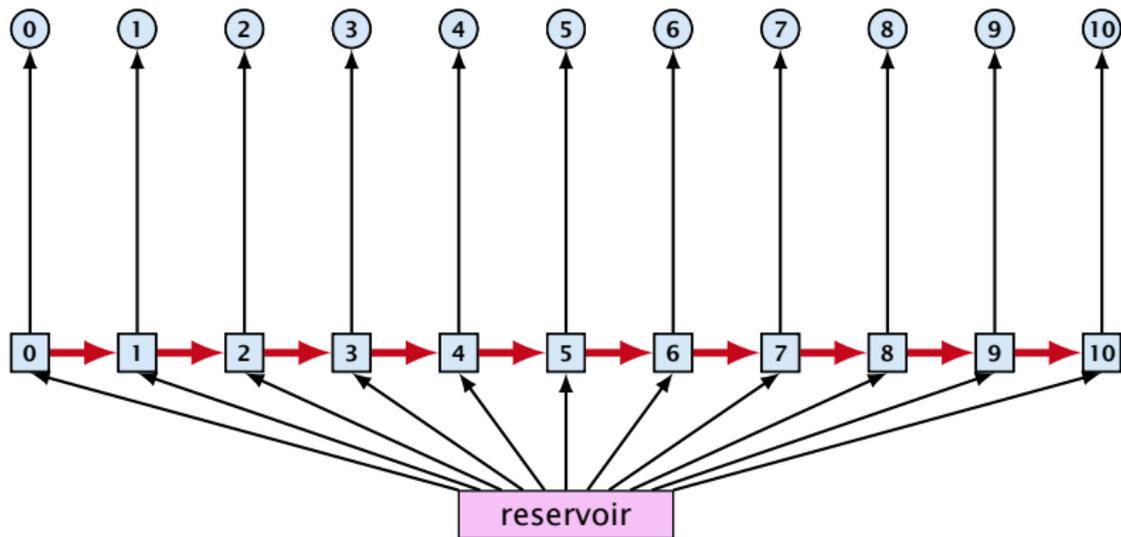
forward edges:

upper bound: $u(e_i) = \infty$;
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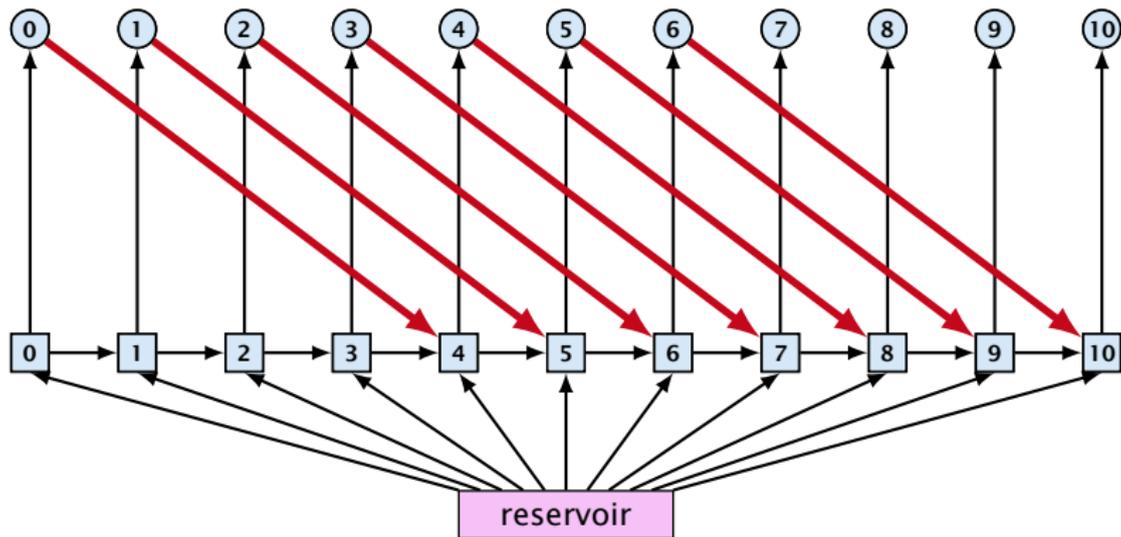
slow edges:

upper bound: $u(e_i) = \infty$;
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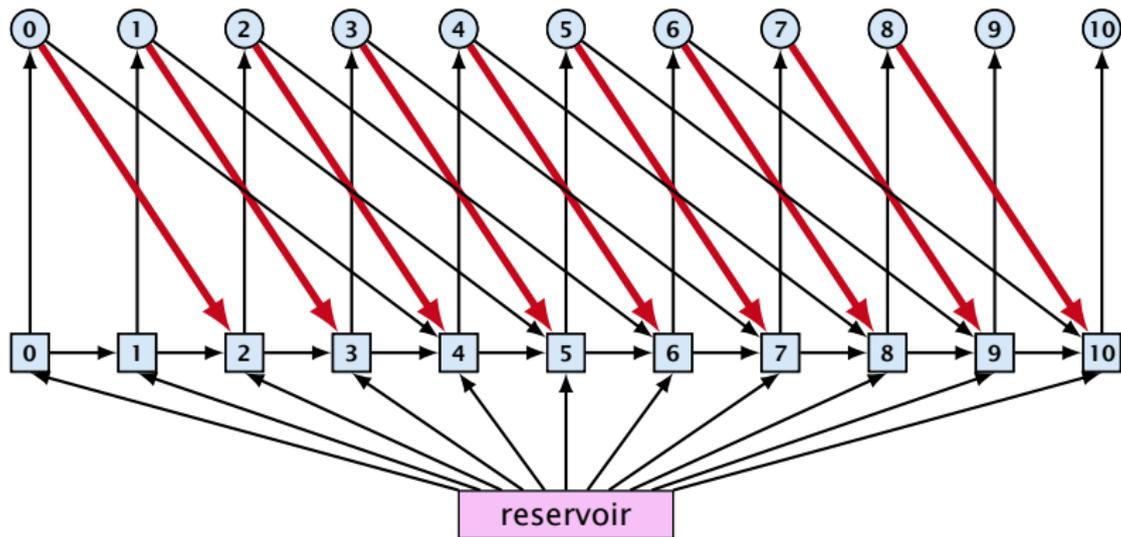
fast edges:

| upper bound: $u(e_i) = \infty$; |
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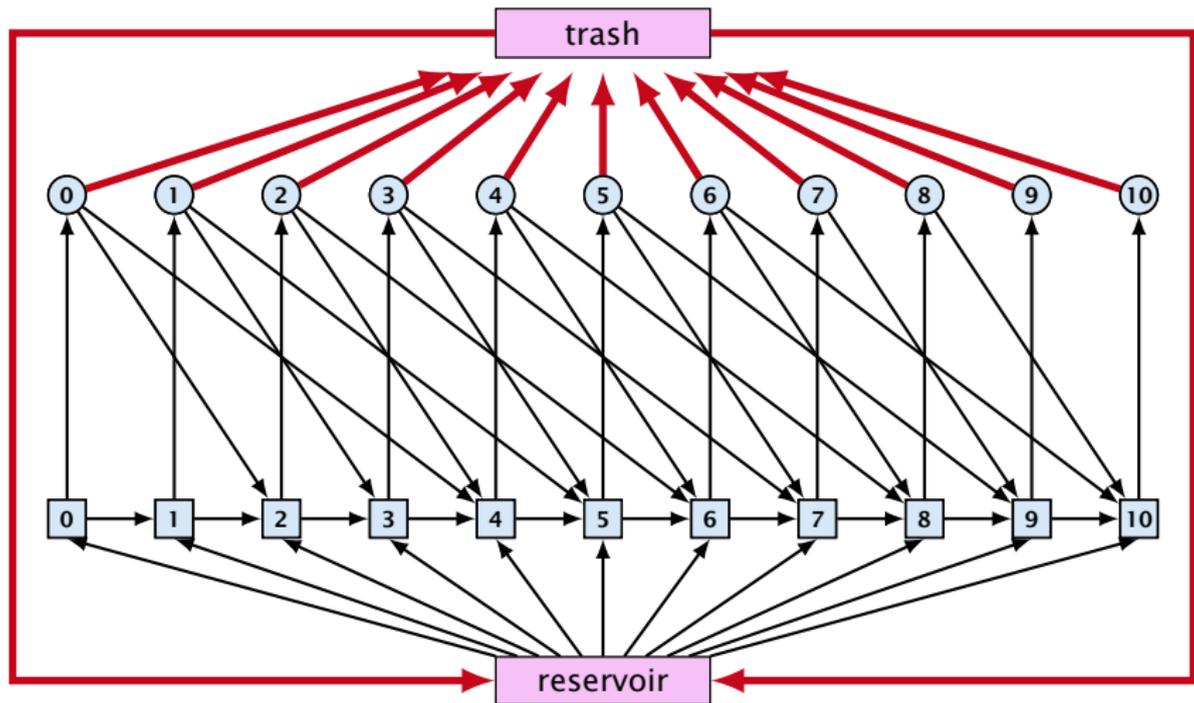
trash edges:

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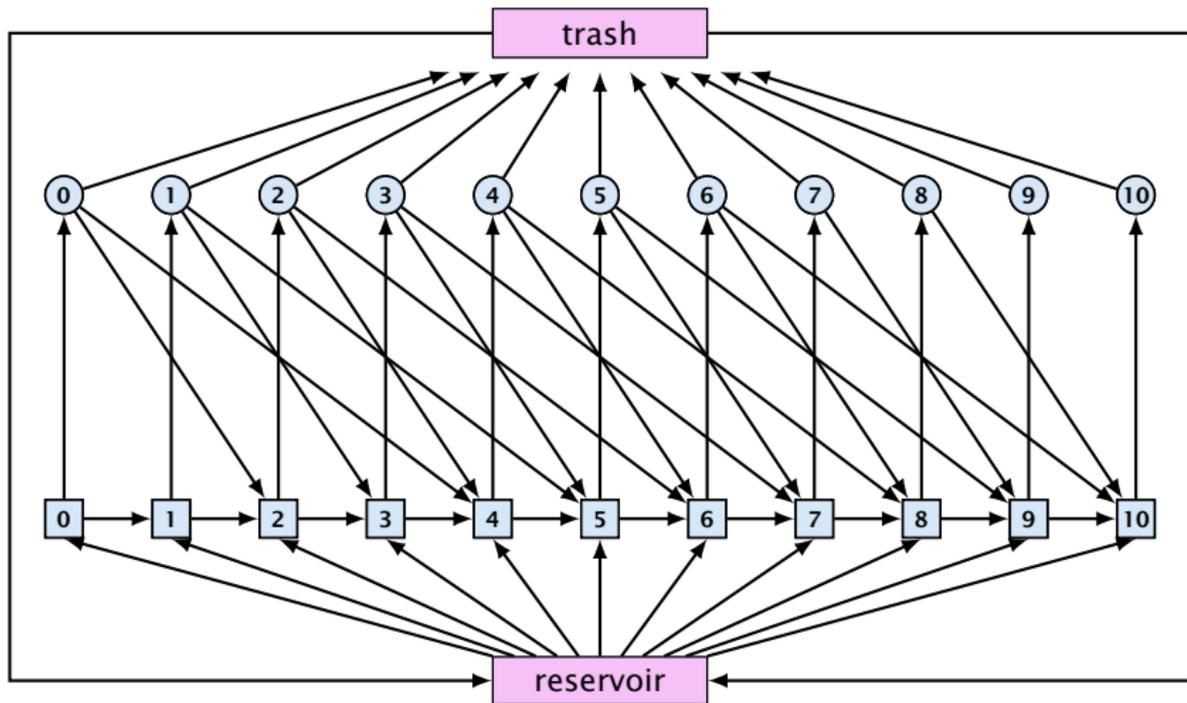
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A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

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Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

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▶ If this path has negative cost you are done.

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Algorithm 51 CycleCanceling($G = (V, E), c, u, b$)

1: establish a feasible flow f in G
2: **while** G_f contains negative cycle **do**
3:     use Bellman-Ford to find a negative circuit Z
4:     $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
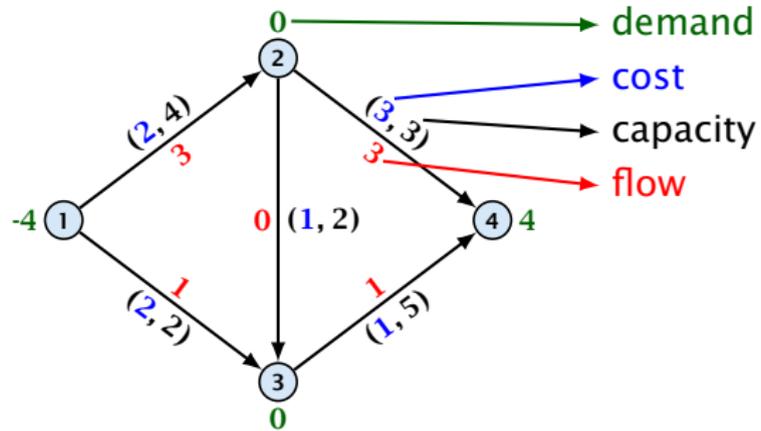5:   augment δ units along Z and update G_f
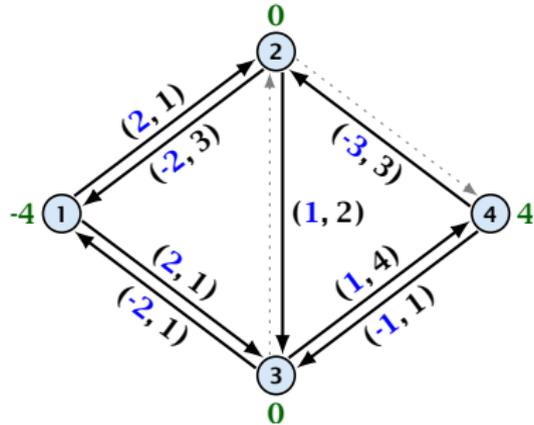
How do we find the initial feasible flow?
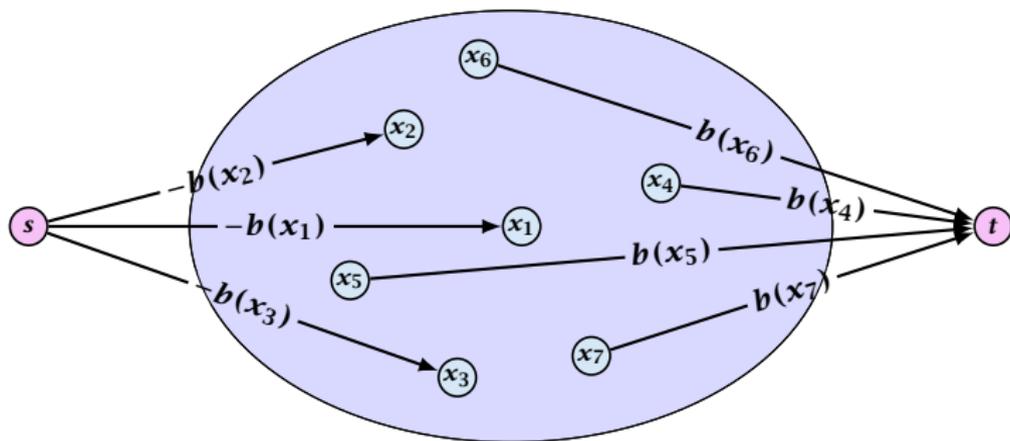
- ▶ Connect new node s to all nodes with negative $b(v)$-value.
- ▶ Connect nodes with positive $b(v)$-value to a new node t.
- ▶ There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

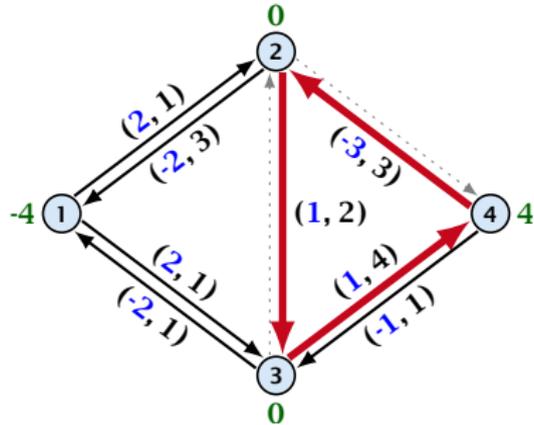$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) \ .$$
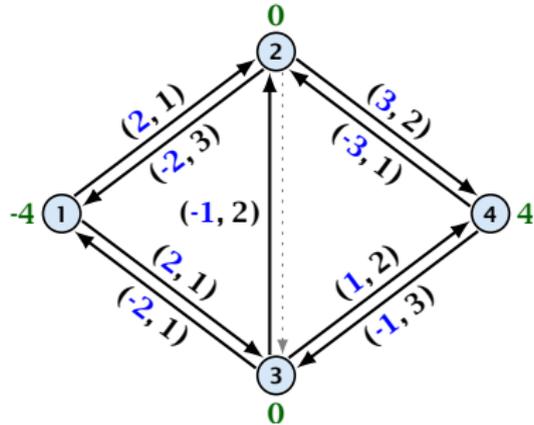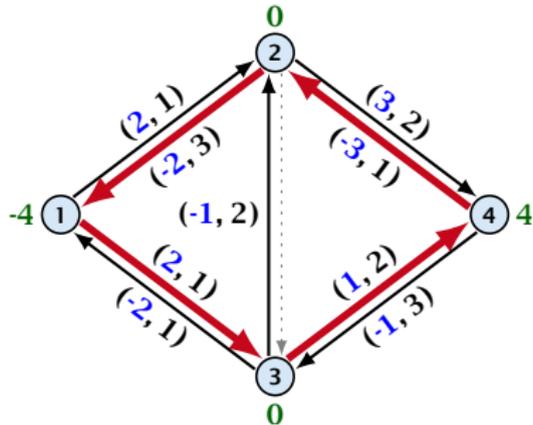
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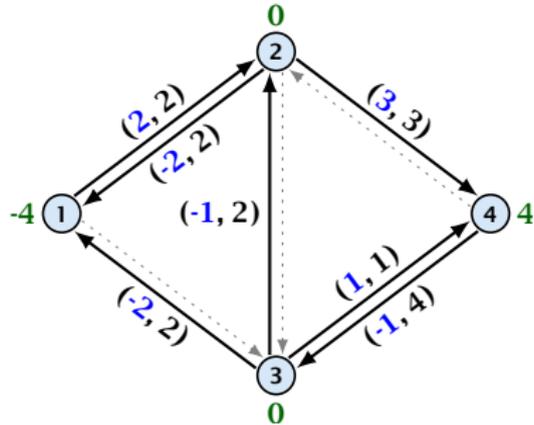
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Lemma 48

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e, $|c(e)| \le C$ and $|u(e)| \le U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-mCU, \ldots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

15 Mincost Flow

A general mincost flow problem is of the following form:

$$\begin{aligned}\min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \ a(v) \le f(v) \le b(v)\end{aligned}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$
$c : E \to \mathbb{R}$;

Lemma 49 (without proof)

A general mincost flow problem can be solved in polynomial time.