6.4 Generating Functions

Definition 4 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

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$$F(z) := \sum_{n\geq 0} a_n z^n ;$$

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There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

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We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

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It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\geq 0}^{\infty} z^n\right) = 1 \ .$$

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

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5. Write $f(z)$ as a formal power series.
   Techniques:
   - partial fraction decomposition (Partialbruchzerlegung)
   - lookup in tables
6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}, a_0 = 1$

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&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
\end{aligned}$$

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$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

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This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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   This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

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   This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
$$A + 3B = 1$$

   which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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$$= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1) z^n$$

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$$= \sum_{n\geq 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\right) z^n$$

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&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big)z^n \\
&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n
\end{aligned}
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&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n
\end{aligned}
$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.