Searching

An extension of binary search with p processors gives that one can find the rank of an element in

$$\log_{p+1}(n) = \frac{\log n}{\log(p+1)}$$

many parallel steps with p processors. (not work-optimal)

This requires a CREW PRAM model. For the EREW model searching cannot be done faster than $\mathcal{O}(\log n - \log p)$ with p processors even if there are p copies of the search key.

Merging

Given two sorted sequences $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, compute the sorted squence $C = (c_1, \dots, c_n)$.

Definition 1
Let $X = (x_1, \dots, x_t)$ be a sequence. The rank $\mathrm{rank}(y : X)$ of y in X is

$$\mathrm{rank}(y : X) = |\{x \in X \mid x \le y\}|$$

For a sequence $Y = (y_1, \dots, y_s)$ we define $\mathrm{rank}(Y : X) := (r_1, \dots, r_s)$ with $r_i = \mathrm{rank}(y_i : X)$.

Merging

We have already seen a merging-algorithm that runs in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n)$.

Using the fast search algorithm we can improve this to a running time of $\mathcal{O}(\log\log n)$ and work $\mathcal{O}(n \log\log n)$.

Merging

Input: $A = a_1, \dots, a_n$; $B = b_1, \dots, b_m$; $m \le n$

1. if $m < 4$ then rank elements of B, using the parallel search algorithm with p processors. Time: $\mathcal{O}(1)$. Work: $\mathcal{O}(n)$.
2. Concurrently rank elements $b_{\sqrt{m}}, b_{2\sqrt{m}}, \dots, b_m$ in A using the parallel search algorithm with $p = \sqrt{n}$. Time: $\mathcal{O}(1)$. Work: $\mathcal{O}(\sqrt{m} \cdot \sqrt{n}) = \mathcal{O}(n)$

   $$j(i) := \mathrm{rank}(b_{i\sqrt{m}} : A)$$

3. Let $B_i = (b_{i\sqrt{m}+1}, \dots, b_{(i+1)\sqrt{m}-1})$; and $A_i = (a_{j(i)+1}, \dots, a_{j(i+1)})$.

   Recursively compute $\mathrm{rank}(B_i : A_i)$.
4. Let k be index not a multiple of $\sqrt{m}$. $i = \lceil \frac{k}{\sqrt{m}} \rceil$. Then $\mathrm{rank}(b_k : A) = j(i) + \mathrm{rank}(b_k : A_i)$.

The algorithm can be made work-optimal by standard techniques.

proof on board...

Mergesort

Lemma 2
A straightforward parallelization of Mergesort can be implemented in time $\mathcal{O}(\log n \log\log n)$ and with work $\mathcal{O}(n \log n)$.

This assumes the CREW-PRAM model.

Mergesort

Let $L[v]$ denote the (sorted) sublist of elements stored at the leaf nodes rooted at v.

We can view Mergesort as computing $L[v]$ for a complete binary tree where the leaf nodes correspond to nodes in the given array.

Since the merge-operations on one level of the complete binary tree can be performed in parallel we obtain time $\mathcal{O}(h \log\log n)$ and work $\mathcal{O}(hn)$, where $h = \mathcal{O}(\log n)$ is the height of the tree.

Pipelined Mergesort

We again compute $L[v]$ for every node in the complete binary tree.

After round s, $L_s[v]$ is an **approximation** of $L[v]$ that will be improved in future rounds.

For $s \geq 3\,\mathrm{height}(v)$, $L_s[v] = L[v]$.

Pipelined Mergesort

In every round, a node v sends $\text{sample}(L_s[v])$ (an approximation of its current list) upwards, and receives approximations of the lists of its children.

It then computes a new approximation of its list.

A node is called active in round s if $s \le 3\,\text{height}(v)$ (this means its list is not yet complete at the start of the round, i.e., $L_{s-1}[v] \ne L[v]$).

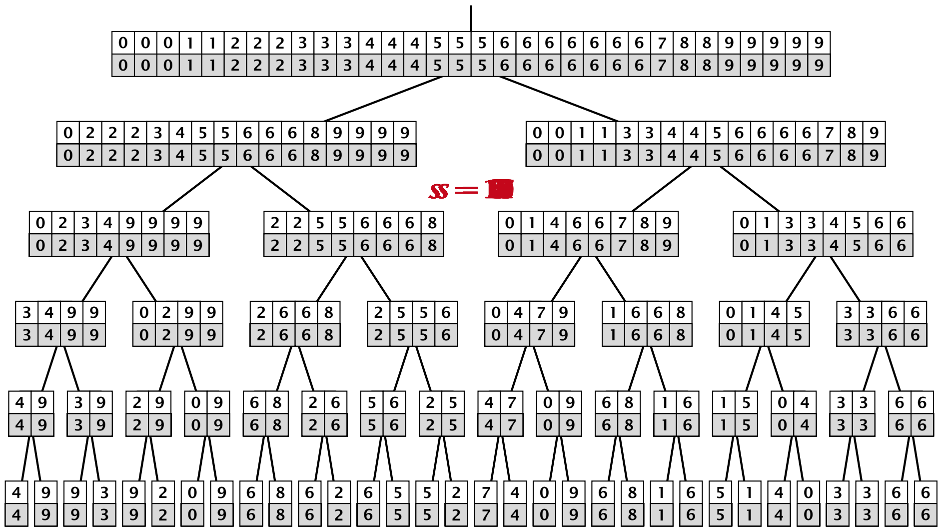
Pipelined Mergesort

Algorithm 11 ColeSort()

```
initialize L_0[v] = A_v for leaf nodes; L_0[v] = ∅ otw.
for s ← 1 to 3 · height(T) do
    for all active nodes v do
        // u and w children of v
        L'_s[u] ← sample(L_{s-1}[u])
        L'_s[w] ← sample(L_{s-1}[w])
        L_s[v] ← merge(L'_s[u], L'_s[w])
```

$$\text{sample}(L_s[v]) = \begin{cases} \text{sample}_4(L_s[v]) & s \le 3\,\text{height}(v) \\ \text{sample}_2(L_s[v]) & s = 3\,\text{height}(v) + 1 \\ \text{sample}_1(L_s[v]) & s = 3\,\text{height}(v) + 2 \end{cases}$$

Colesort



Pipelined Mergesort

Lemma 3
After round $s = 3\,\text{height}(v)$, the list $L_s[v]$ is complete.

Proof:

- clearly true for leaf nodes
- suppose it is true for all nodes up to height h;
- fix a node v on level $h+1$ with children u and w
- $L_{3h}[u]$ and $L_{3h}[w]$ are complete by induction hypothesis
- further $\text{sample}(L_{3h+2}[u]) = L[u]$ and $\text{sample}(L_{3h+2}[v]) = L[v]$
- hence in round $3h+3$ node v will merge the complete list of its children; after the round $L[v]$ will be complete

Pipelined Mergesort

Lemma 4
The number of elements in lists $L_s[v]$ for active nodes v is at most $\mathcal{O}(n)$.

proof on board...

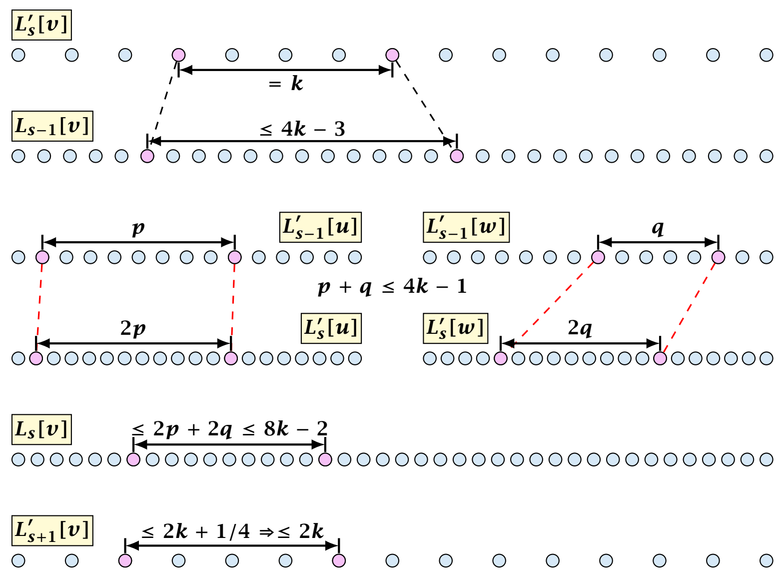
Definition 5
A sequence X is a c-cover of a sequence Y if for any two consecutive elements α, β from $(-\infty, X, \infty)$ the set $|\{y_i \mid \alpha \le y_i \le \beta\}| \le c$.

Pipelined Mergesort

Lemma 6
$L'_s[v]$ is a 4-cover of $L'_{s+1}[v]$.

If $[a, b]$ fulfills $|[a, b] \cap (A \cup \{-\infty, \infty\})| = k$ we say $[a, b]$ intersects $(-\infty, A, +\infty)$ in k items.

Lemma 7
If $[a, b]$ with $a, b \in L'_s[v] \cup \{-\infty, \infty\}$ intersects $(-\infty, L'_s[v], \infty)$ in $k \ge 2$ items, then $[a, b]$ intersects $(-\infty, L'_{s+1}, \infty)$ in at most $2k$ items.



Note that the last step holds as long $L'_{s+1}[v] = \text{sample}_4(L_s[v])$. Otw. $L_{s-1}[v]$ has already been full, and hence, $L'_s[v], L'_{s+1}[v], L'_{s+2}[v]$ are 4-covers of the complete list $L[v]$, and also 4-covers of each other.

Merging with a Cover

Lemma 8
Given two sorted sequences A and B. Let X be a c-cover of A and B for constant c, and let $\mathrm{rank}(X:A)$ and $\mathrm{rank}(X:B)$ be known.

We can merge A and B in time $\mathcal{O}(1)$ using $\mathcal{O}(|X|)$ operations.

Merging with a Cover

Lemma 9
Given two sorted sequences A and B. Let X be a c-cover of B for constant c, and let $\mathrm{rank}(A:X)$ and $\mathrm{rank}(X:B)$ be known.

We can compute $\mathrm{rank}(A:B)$ using $\mathcal{O}(|X|+|A|)$ operations.

Merging with a Cover

Lemma 10
Given two sorted sequences A and B. Let X be a c-cover of B for constant c, and let $\mathrm{rank}(A:X)$ and $\mathrm{rank}(X:B)$ be known.

We can compute $\mathrm{rank}(B:A)$ using $\mathcal{O}(|X|+|A|)$ operations.

Easy to do with concurrent read. Can also be done with exclusive read but non-trivial.

In order to do the merge in iteration $s+1$ in constant time we need to know

$$\mathrm{rank}(L_s[v]:L'_{s+1}[u]) \text{ and } \mathrm{rank}(L_s[v]:L'_{s+1}[w])$$

and we need to know that $L_s[v]$ is a 4-cover of $L'_{s+1}[u]$ and $L'_{s+1}[w]$.

Lemma 11

$L_s[v]$ is a 4-cover of $L'_{s+1}[u]$ and $L'_{s+1}[w]$.

- $L_s[v] \supseteq L'_s[u], L'_s[w]$
- $L'_s[u]$ is 4-cover of $L'_{s+1}[u]$
- Hence, $L_s[v]$ is 4-cover of $L'_{s+1}[u]$ as adding more elements cannot destroy the cover-property.

Analysis

Lemma 12

Suppose we know for every internal node v with children u and w

- $\text{rank}(L'_s[v] : L'_{s+1}[v])$
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_s[u])$

We can compute

- $\text{rank}(L'_{s+1}[v] : L'_{s+2}[v])$
- $\text{rank}(L'_{s+1}[u] : L'_{s+1}[w])$
- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$

in constant time and $\mathcal{O}(|L_{s+1}[v]|)$ operations, where v is the parent of u and w.

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ (**4-cover**)
- $\text{rank}(L'_s[w] : L'_s[u])$
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ (**4-cover**)

Compute

- $\text{rank}(L'_{s+1}[w] : L'_s[u])$
- $\text{rank}(L'_{s+1}[u] : L'_s[w])$

Compute

- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$
- $\text{rank}(L'_{s+1}[u] : L'_{s+1}[w])$

ranks between siblings can be computed easily

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ (**4-cover** $\rightarrow \text{rank}(L'_{s+1}[u] : L'_s[u])$)
- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
- $\text{rank}(L'_s[u] : L'_{s+1}[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ (**4-cover** $\rightarrow \text{rank}(L'_{s+1}[w] : L'_s[w])$)

Compute (recall that $L_s[v] = \text{merge}(L'_s[u], L'_s[w])$)

- $\text{rank}(L_s[v] : L'_{s+1}[u])$
- $\text{rank}(L_s[v] : L'_{s+1}[w])$

Compute

- $\text{rank}(L_s[v] : L_{s+1}[v])$ (by adding)
- $\text{rank}(L'_{s+1}[v] : L'_{s+2}[v])$ (by sampling)