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Randomized Algorithms

Exercise Sheet 2

Due: October 27, 2014

Exercise 1 (5 points)
Consider the randomized min cut algorithm presented in class. We showed that, for any graph G with n vertices, the probability that this algorithm generates a specific min cut C of G is at least $2/(n(n-1))$. By using this fact, what can we say about the maximum number of distinct min cuts that a graph G can have?

Exercise 2 (5 points)
Recall the min (s,t)-cut problem in which we are given an undirected connected multigraph $G = (V, E)$ with two distinguished vertices $s, t \in V$ and the objective is to find an (s,t)-cut of minimum size, where an (s,t)-cut is a subset of edges $C \subseteq E$ whose removal from G disconnects s from t. Find an example of a graph in which the number of distinct min (s,t)-cuts is 2^n.

Exercise 3 (10 points)
Given a set of elements $A_n = \{1, 2, \ldots, n\}$, propose a polynomial-time algorithm for generating a random permutation of A_n uniformly among all possible permutations and show that your algorithm is correct. You may assume that one computational step suffices for choosing an integer uniformly at random in the range $\{1, 2, \ldots, k\}$, for any positive integer $k \leq n$.

Exercise 4 (10 points)
Consider a permutation π of the set of elements $A_n = \{1, 2, \ldots, n\}$ chosen uniformly at random among all possible permutations and let $\pi(i)$ be the position of the element i in π. By using the permutation π, we may construct a directed graph G as follows. We introduce a vertex for each $i \in A_n$. Moreover, for each $i \in A_n$, we add a directed arc from i to $\pi(i)$ indicating that the element i is in the position $\pi(i)$.

- Show that G can be decomposed into a set of disjoint cycles $C_1, C_2, \ldots, C_\ell$ such that every vertex and every arc of G belong to exactly one of these cycles.
- For each $i \in A_n$, we define a random variable $X_i = \frac{1}{k}$, where k is the length of the cycle containing i. Show that $Pr(X_i = \frac{1}{k}) = \frac{1}{n}$, for any $k \in A_n$.
- What is the expected number of cycles in G?

Exercise 5 (10 points)
Consider the Binary Planar Partition problem presented in class. Recall that the input is a set of n non-intersecting line segments in the plane and the objective is to find a binary planar partition (or simply partition) such that every region in the partition contains at most one line segment (or a portion of one line segment). The size of a partition is the number of regions it contains. For each of the following situations, give an example of a set of non-intersecting line segments in the plane such that the situation occurs:

- The (worst-case) partition constructed by the randomized algorithm presented in class has size $\Omega(n^2)$, while there exists a partition of size $O(n)$.
- No partition can avoid breaking some line segments into pieces.
- There exists a partition of size n, whereas any auto-partition has size at least $\frac{4n}{3}$.