

# Strong Duality

## Theorem 2 (Strong Duality)

Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z^*$  and  $w^*$  denote the optimal solution to  $P$  and  $D$ , respectively.

Then

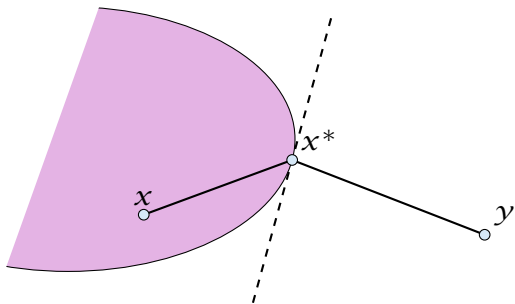
$$z^* = w^*$$

### Lemma 3 (Weierstrass)

Let  $X$  be a compact set and let  $f(x)$  be a continuous function on  $X$ . Then  $\min\{f(x) : x \in X\}$  exists.

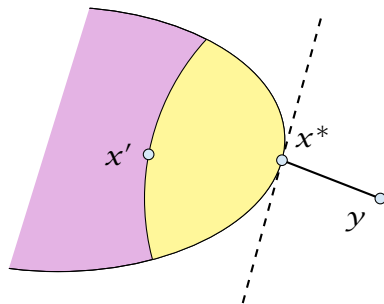
## Lemma 4 (Projection Lemma)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty convex set, and let  $y \notin X$ . Then there exist  $x^* \in X$  with minimum distance from  $y$ . Moreover for all  $x \in X$  we have  $(y - x^*)^T (x - x^*) \leq 0$ .



# Proof of the Projection Lemma

- ▶ Define  $f(x) = \|y - x\|$ .
- ▶ We want to apply Weierstrass but  $X$  may not be bounded.
- ▶  $X \neq \emptyset$ . Hence, there exists  $x' \in X$ .
- ▶ Define  $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$ . This set is closed and bounded.
- ▶ Applying Weierstrass gives the existence.



## Proof of the Projection Lemma (continued)

$x^*$  is minimum. Hence  $\|y - x^*\|^2 \leq \|y - x\|^2$  for all  $x \in X$ .

By **convexity**:  $x \in X$  then  $x^* + \epsilon(x - x^*) \in X$  for all  $0 \leq \epsilon \leq 1$ .

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

Hence,  $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\epsilon \|x - x^*\|^2$ .

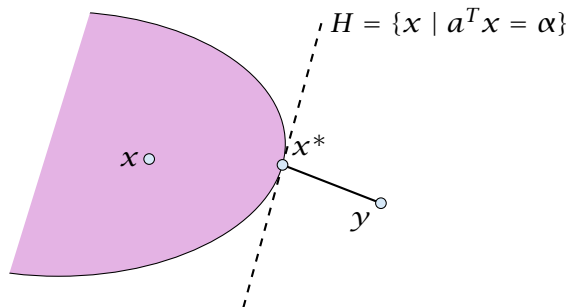
Letting  $\epsilon \rightarrow 0$  gives the result.

## Theorem 5 (Separating Hyperplane)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty closed convex set, and let  $y \notin X$ . Then there exists a *separating hyperplane*  $\{x \in \mathbb{R}^m : a^T x = \alpha\}$  where  $a \in \mathbb{R}^m$ ,  $\alpha \in \mathbb{R}$  that *separates*  $y$  from  $X$ . ( $a^T y < \alpha$ ;  $a^T x \geq \alpha$  for all  $x \in X$ )

## Proof of the Hyperplane Lemma

- ▶ Let  $x^* \in X$  be closest point to  $y$  in  $X$ .
- ▶ By previous lemma  $(y - x^*)^T(x - x^*) \leq 0$  for all  $x \in X$ .
- ▶ Choose  $a = (x^* - y)$  and  $\alpha = a^T x^*$ .
- ▶ For  $x \in X$ :  $a^T(x - x^*) \geq 0$ , and, hence,  $a^T x \geq \alpha$ .
- ▶ Also,  $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$



## Lemma 6 (Farkas Lemma)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then *exactly one* of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax = b$ ,  $x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^T y \geq 0$ ,  $b^T y < 0$

Assume  $\hat{x}$  satisfies 1. and  $\hat{y}$  satisfies 2. Then

$$0 > \hat{y}^T b = \hat{y}^T A \hat{x} \geq 0$$

Hence, at most one of the statements can hold.



## Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider  $S = \{Ax : x \geq 0\}$  so that  $S$  closed, convex,  $b \notin S$ .

We want to show that there is  $y$  with  $A^T y \geq 0$ ,  $b^T y < 0$ .

Let  $y$  be a hyperplane that separates  $b$  from  $S$ . Hence,  $y^T b < \alpha$  and  $y^T s \geq \alpha$  for all  $s \in S$ .

$$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$$

$y^T Ax \geq \alpha$  for all  $x \geq 0$ . Hence,  $y^T A \geq 0$  as we can choose  $x$  arbitrarily large.

## Lemma 7 (Farkas Lemma; different version)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then exactly one of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax \leq b$ ,  $x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^T y \geq 0$ ,  $b^T y < 0$ ,  $y \geq 0$

**Rewrite the conditions:**

1.  $\exists x \in \mathbb{R}^n$  with  $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$ ,  $x \geq 0$ ,  $s \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $\begin{bmatrix} A^T \\ I \end{bmatrix} y \geq 0$ ,  $b^T y < 0$

# Proof of Strong Duality

$$P: z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$D: w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

## Theorem 8 (Strong Duality)

Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z$  and  $w$  denote the optimal solution to  $P$  and  $D$ , respectively (i.e.,  $P$  and  $D$  are non-empty). Then

$$z = w .$$

# Proof of Strong Duality

$z \leq w$ : follows from weak duality

$z \geq w$ :

We show  $z < \alpha$  implies  $w < \alpha$ .

$$\exists x \in \mathbb{R}^n$$

$$\begin{aligned} \text{s.t.} \quad Ax &\leq b \\ -c^T x &\leq -\alpha \\ x &\geq 0 \end{aligned}$$

$$\exists y \in \mathbb{R}^m; v \in \mathbb{R}$$

$$\begin{aligned} \text{s.t.} \quad A^T y - cv &\geq 0 \\ b^T y - \alpha v &< 0 \\ y, v &\geq 0 \end{aligned}$$

From the definition of  $\alpha$  we know that the first system is infeasible; hence the second must be feasible.

# Proof of Strong Duality

$$\begin{array}{ll} \exists \mathbf{y} \in \mathbb{R}^m; \mathbf{v} \in \mathbb{R} & \\ \text{s.t.} & A^T \mathbf{y} - \mathbf{v} \geq 0 \\ & \mathbf{b}^T \mathbf{y} - \alpha \mathbf{v} < 0 \\ & \mathbf{y}, \mathbf{v} \geq 0 \end{array}$$

If the solution  $\mathbf{y}, \mathbf{v}$  has  $\mathbf{v} = 0$  we have that

$$\begin{array}{ll} \exists \mathbf{y} \in \mathbb{R}^m & \\ \text{s.t.} & A^T \mathbf{y} \geq 0 \\ & \mathbf{b}^T \mathbf{y} < 0 \\ & \mathbf{y} \geq 0 \end{array}$$

is feasible. By Farkas lemma this gives that LP  $P$  is infeasible.  
Contradiction to the assumption of the lemma.

# Proof of Strong Duality

Hence, there exists a solution  $y, v$  with  $v > 0$ .

We can rescale this solution (scaling both  $y$  and  $v$ ) s.t.  $v = 1$ .

Then  $y$  is feasible for the dual but  $b^T y < \alpha$ . This means that  $w < \alpha$ .

# Fundamental Questions

## Definition 9 (Linear Programming Problem (LP))

Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ ,  $c \in \mathbb{Q}^n$ ,  $\alpha \in \mathbb{Q}$ . Does there exist  $x \in \mathbb{Q}^n$  s.t.  $Ax = b$ ,  $x \geq 0$ ,  $c^T x \geq \alpha$ ?

### Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP? yes!
- ▶ Is LP in P?

### Proof:

- ▶ Given a primal maximization problem  $P$  and a parameter  $\alpha$ . Suppose that  $\alpha > \text{opt}(P)$ .
- ▶ We can prove this by providing an optimal basis for the dual.
- ▶ A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost  $< \alpha$ .