

18 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

- ▶ Input: undirected, bipartite graph $G = L \cup R, E$.
- ▶ an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- ▶ find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

Simplifying Assumptions (wlog [why?]):

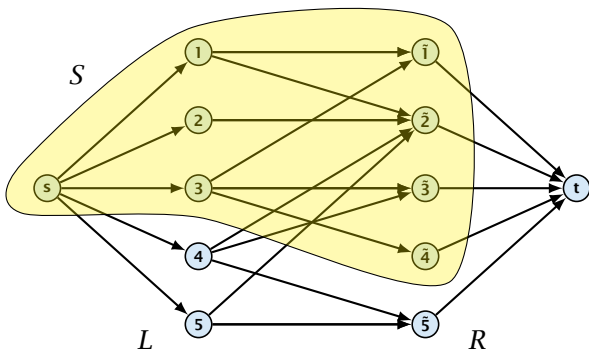
- ▶ assume that $|L| = |R| = n$
- ▶ assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$
- ▶ can assume goal is to construct maximum weight **perfect** matching

Weighted Bipartite Matching

Theorem 1 (Halls Theorem)

A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in R that have a neighbour in S .

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Halls Theorem

Proof:

- ← Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neighbours.
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 - ▶ Let S denote a minimum cut and let $L_S \cong L \cap S$ and $R_S \cong R \cap S$ denote the portion of S inside L and R , respectively.
 - ▶ Clearly, all neighbours of nodes in L_S have to be in S , as otherwise we would cut an edge of infinite capacity.
 - ▶ This gives $R_S \geq |\Gamma(L_S)|$.
 - ▶ The size of the cut is $|L| - |L_S| + |R_S|$.
 - ▶ Using the fact that $|\Gamma(L_S)| \geq |L_S|$ gives that this is at least $|L|$.

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- ▶ Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

- ▶ Let $H(\vec{x})$ denote the subgraph of G that only contains edges that are **tight** w.r.t. the node weighting \vec{x} , i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.
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Reason:

- ▶ The weight of your matching M^* is

$$\sum_{(u,v) \in M^*} w_{(u,v)} = \sum_{(u,v) \in M^*} (x_u + x_v) = \sum_v x_v .$$

- ▶ Any other perfect matching M (in G , not necessarily in $H(\vec{x})$) has

$$\sum_{(u,v) \in M} w_{(u,v)} \leq \sum_{(u,v) \in M} (x_u + x_v) = \sum_v x_v .$$

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What if you don't find a perfect matching?

Then, Hall's theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

Idea: reweight such that:

- ▶ the total weight assigned to nodes decreases
- ▶ the weight function still dominates the edge-weights

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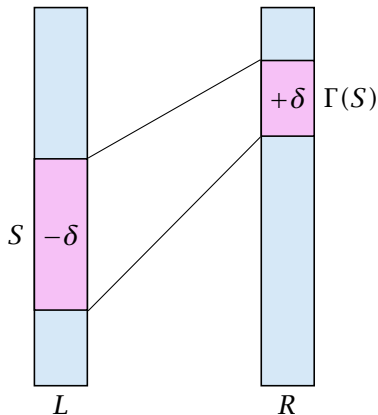
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Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.

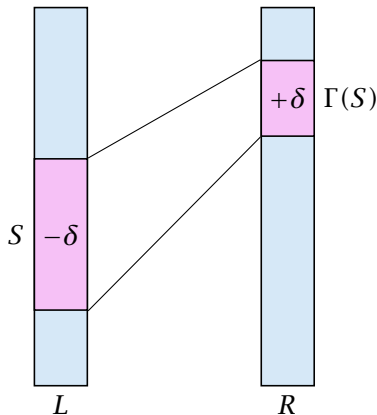
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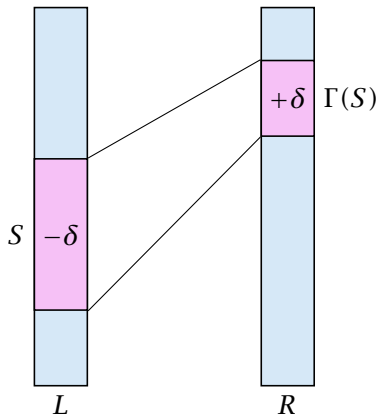
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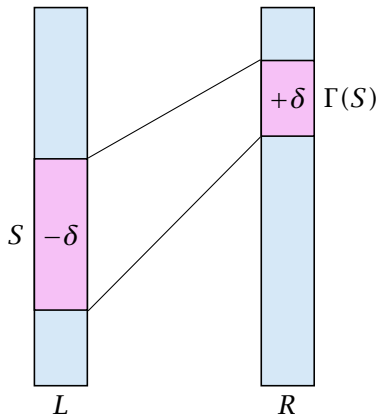
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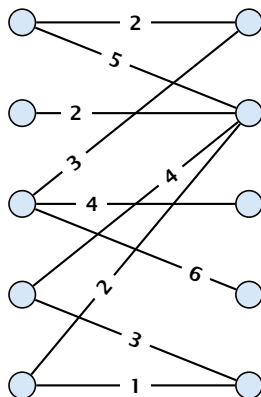
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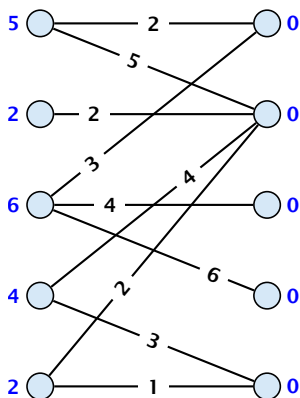
Weighted Bipartite Matching

Edges not drawn have weight 0.



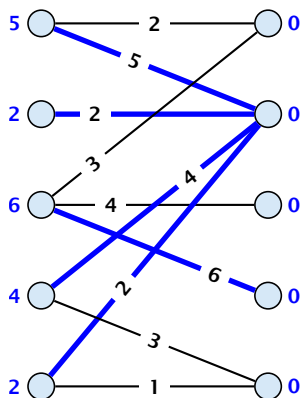
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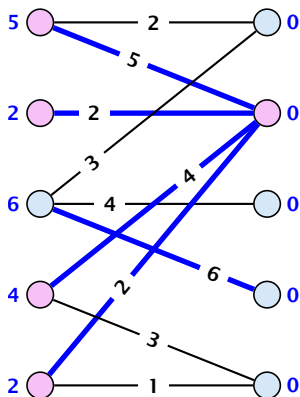
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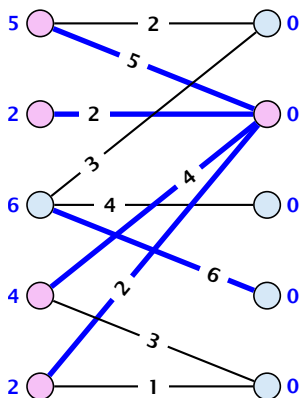
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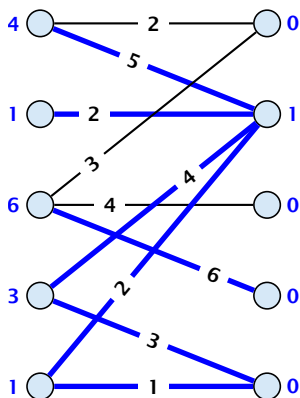
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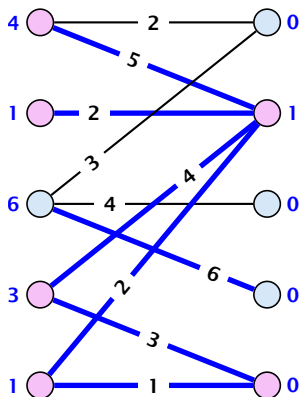
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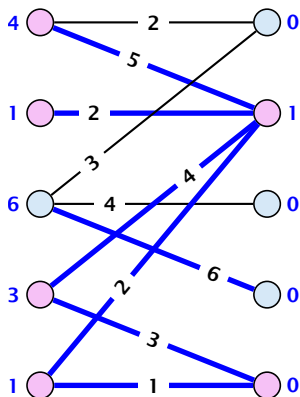
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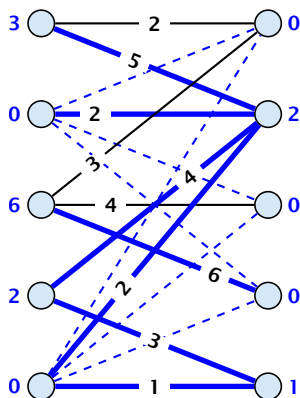
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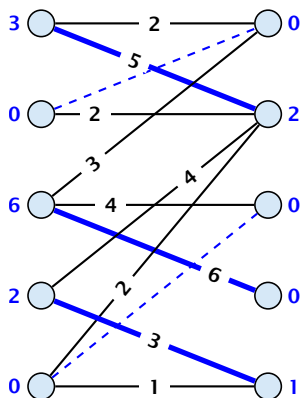
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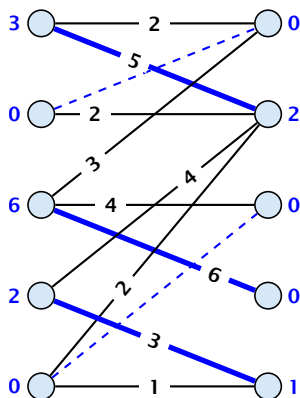
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How many iterations do we need?

- ▶ One reweighting step increases the number of edges out of S by at least one.
- ▶ Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in S (we will show that we can always find S and a matching such that this holds).
- ▶ This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and S or between $L - S$ and $R - \Gamma(S)$.
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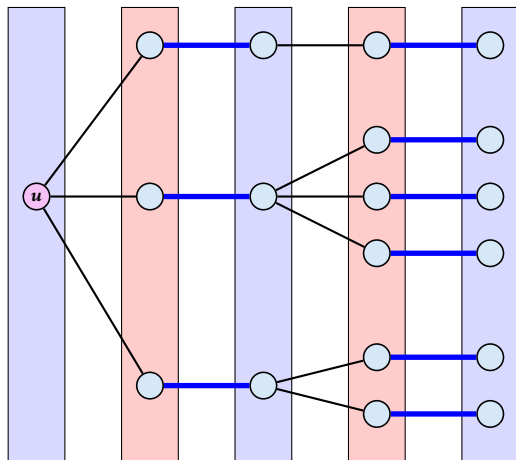
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- ▶ We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.
- ▶ This gives a polynomial running time.

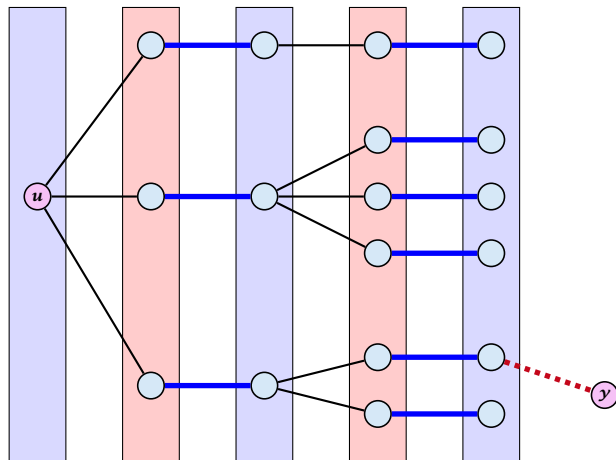
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Construct an alternating tree.



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How do we find S ?

- ▶ Start on the left and compute an alternating tree, starting at any free node u .
- ▶ If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).
- ▶ The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.
- ▶ All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex u . Hence, $|V_{\text{odd}}| = |E(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

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- ▶ The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).
- ▶ After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd} . After at most n reweights we can do an augmentation.
- ▶ A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).
- ▶ An augmentation takes at most $\mathcal{O}(n)$ time.
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