Fundamental Algorithms 4

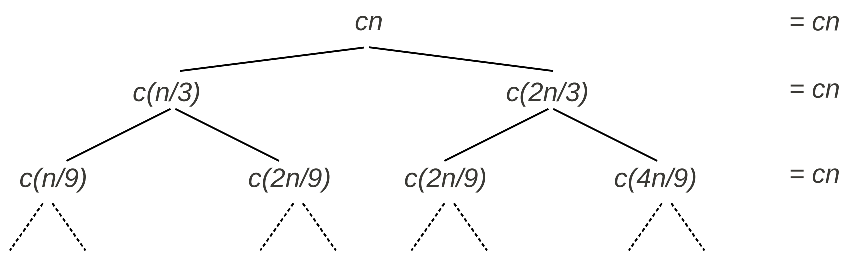
Exercise 1

Try the Recursion Tree Method (compare lecture) for the following recurrence:

$$T(n) = T(n/3) + T(2n/3) + O(n)$$

Show that the height of the recursion tree is in $O(\log(n))$.

- We assume that all occurring n are multiples of 3. Further, let c be the constant in the $O(n)$ term. We then obtain the recursion tree



  On each level, we obviously obtain cn operations, independent of the level.

- The longest path in the recursion tree is the rightmost path with problem size $n \to 2/3n \to (2/3)^2 n \to \cdots \to 1$ until we stop at problem size 1. The height h of the tree can be determined via the equation $(2/3)^h n = 1$, leading to $h = \log_{3/2} n$.

  We could expect the total cost to be $O(cn \log_{3/2} n) = O(n \log n)$.

What could be a flaw using the recursion tree method for such unbalanced trees?
Show that $T(n) \in O(n \log(n))$, anyway, by using the substitution method.

- Problem: If the tree was a complete binary tree, we would have $2^{\log_{3/2} n} = n^{\log_{3/2} 2}$ leaves (as $\log_{3/2} n = \log_2 n / \log_2 \frac{3}{2} = \log_2 n / \log_{\frac{3}{2}} 2$, using the formula $\log_a b = 1/\log_b a$). As $\log_{3/2} 2 > 1$, the number of terms would be $\omega(n \log n)$ on the last level. Hence, the simple approach of assuming constant effort c for T(1) on the final level does no longer work: in that case, the costs would sum up to $\Theta(cn^{\log_{3/2} 2})$ on the last level – and not cn!

Hence, we'd have to explicitly consider that the tree starts to thin out much earlier (starting at level $1 + \log_3 n$), and we would have to examine the exact cost on all subsequent levels, which is more tedious than our tree diagram suggests.

- We simplify and assume that the total cost are $O(n \log n)$ and use the substitution method to verify this:

  Assuming that $T(n) \leq an \log n$ for a suitable constant a, we obtain

$$
\begin{aligned}
T(n) &\leq T(n/3) + T(2n/3) + cn \\
&\leq a(n/3)\log(n/3) + a(2n/3)\log(2n/3) + cn \\
&= a3n/3\log n - a\left((n/3)\log 3 + (2n/3)\log(3/2)\right) + cn \\
&= an\log n - a\left((n/3)\log 3 + (2n/3)\log 3 - (2n/3)\log 2\right) + cn \\
&= an\log n - an\left(\log 3 - 2/3\log 2\right) + cn \\
&\leq an\log n
\end{aligned}
$$

  for $d \geq c/(\log 3 - 2/3 \log 2)$.

Exercise 2

For the so-called BFPRT Algorithm, an algorithm to determine the *median* element of an array, we obtain the following (slightly simplified) recurrence equation for its running time $T(n)$ (depending on the number n of elements):

$$T(n) = s\,(n,k) + T\left(\frac{n}{k}\right) + T\left(\frac{l}{2k}n\right).$$

k and l are parameters (k usually small, for example $k = 3$ or $k = 5$) where $k = 2l + 1$. For the function s, we can assume $s(n,k) \in \Theta(n \log k)$.

a) Show that $T(n) \in O(n)$.

b) Does it make sense to use large values for k (and l, resp.)?

Solution:

We try to prove the claim by inserting the assumed solution $T(n) \leq cn$ into the recurrence equation:

$$
\begin{aligned}
cn &\geq s(n,k) + c \cdot \frac{n}{k} + c \cdot \frac{l}{2k}n \\
\Leftrightarrow c\left(n - \frac{n}{k} - \frac{l}{2k}n\right) &\geq s(n,k)
\end{aligned}
$$

As $s(n,k) \in \Theta(n \log k)$, there is a constant C_s such that $s(n,k) \leq C_s n \log k$ for large enough n. Therefore, c has to be large enough to satisfy

$$
\begin{aligned}
c\left(n - \frac{n}{k} - \frac{l}{2k}n\right) &\geq C_s n \log k \geq s(n,k) \\
\Leftrightarrow c &\geq \frac{C_s \log k}{1 - \frac{1}{k} - \frac{l}{2k}} \in O(\log k)
\end{aligned}
$$

Hence, we can choose a suitable, large enough c that is independent of n, and thus prove $T(n) \in O(n)$, but the involved constant has to slightly grow with k, as $c \in O(\log k)$. As a consequence, k should be of limited size.