Fundamental Algorithms

Chapter 2: Sorting

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Part I

Simple Sorts

The Sorting Problem

Definition

Sorting is required to order a given sequence of elements, or more precisely:

Input : a sequence of n elements $a_1, a_2, \ldots, a_n$

Output : a permutation (reordering) $a'_1, a'_2, \ldots, a'_n$ of the input sequence, such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

- we will assume the elements $a_1, a_2, \ldots, a_n$ to be integers (or any element/data type on which a total order $\leq$ is defined)
- a sorting algorithm may output the permuted data or also the permuted set of indices

Insertion Sort

Idea: sorting by inserting

- successively generate ordered sequences of the first j numbers: $j = 1, j = 2, \ldots, j = n$
- in each step, $j \to j + 1$, one additional integer has to be inserted into an already ordered sequence

Data Structures:

- an array A[1..n] that contains the sequence a_1 (in A[1]), . . . , a_n (in A[n]).
- numbers are sorted **in place**:
  output sequence will be stored in A itself
  (hence, content of A is changed)

Insertion Sort – Implementation

```
InsertionSort(A:Array[1..n]) {

  for j from 2 to n {
  // insert A[j] into sequence A[1..j−1]

     key := A[j];

     i := j−1; // initialize i for while loop
     while i>=1 and A[i]>key {
        A[i+1] := A[i];
        i := i−1;
     }
     A[i+1] := key;
  }
}
```

Correctness of InsertionSort

Loop invariant:

Before each iteration of the for-loop, the subarray A[1..j-1] consists of all elements originally in A[1..j-1], but in sorted order.

Initialization:

- loops starts with j=2;
  hence, A[1..j-1] consists of the element A[1] only
- A[1] contains only one element, A[1], and is therefore sorted.

Correctness of InsertionSort

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Before each iteration of the for-loop, the subarray A[1..j-1] consists of all elements originally in A[1..j-1], but in sorted order.

Maintenance:

- assume that the while loop works correctly
  (or prove this using an additional loop invariant):
  - after the while loop, i contains the largest index for which A[i] is smaller than the key
  - A[i+2..j] contains the (sorted) elements previously stored in A[i+1..j-1]; also: A[i+1] and all elements in A[i+2..j] are $\geq$ key
- the key value, A[j], is thus correctly inserted as element A[i+1] (overwrites the duplicate value A[i+1])
- after execution of the loop body, A[1..j] is sorted
- thus, before the next iteration (j:=j+1), A[1..j-1] is sorted

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Before each iteration of the for-loop, the subarray A[1..j-1] consists of all elements originally in A[1..j-1], but in sorted order.

Termination:

- The for-loop terminates when j exceeds n (i.e., j=n+1)
- Thus, at termination, A[1 .. (n+1)-1] = A[1..n] is sorted and contains all original elements

Insertion Sort – Number of Comparisons

```
InsertionSort(A:Array[1..n]) {
    for j from 2 to n {
        key := A[j];
        i := j-1;
        while i>=1 and A[i]>key {
            A[i+1] := A[i];
            i := i-1;
        }
        A[i+1] := key;
    }
}
```

n-1 iterations

t_j iterations
$\rightarrow$ t_j comparisons
A[i] $>$ key

$\Rightarrow \sum_{j=2}^{n} t_j$ comparisons

Insertion Sort – Number of Comparisons (2)

- counted number of comparisons: $T_{\mathrm{IS}} = \sum_{j=2}^{n} t_j$
- where t_j is the number of iterations of the while loop (which is, of course, unknown)
- good estimate for the run time, if the comparison is the most expensive operation (note: replace "i$>$=1" by for loop)

Analysis

- what is the "best case"?
- what is the "worst case"?

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Analysis of the "best case":

- in the best case, $t_j = 1$ for all j
- happens only, if A[1..n] is already sorted

$$\Rightarrow T_{\text{IS}}(n) = \sum_{j=2}^{n} 1 = n - 1 \in \Theta(n)$$

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- good estimate for the run time, if the comparison is the most expensive operation (note: replace "i$>$=1" by for loop)

Analysis of the "worst case":

- in the worst case, $t_j = j - 1$ for all j
- happens, if A[1..n] is already sorted in opposite order

$$\Rightarrow T_{\text{IS}}(n) = \sum_{j=2}^{n} (j-1) = \frac{1}{2} n(n-1) \in \Theta(n^2)$$

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- where t_j is the number of iterations of the while loop (which is, of course, unknown)
- good estimate for the run time, if the comparison is the most expensive operation (note: replace "i$>$=1" by for loop)

Analysis of the "average case":

- best case analysis: $T_{\text{IS}}(n) \in \Theta(n)$
- worst case analysis: $T_{\text{IS}}(n) \in \Theta(n^2)$

$\Rightarrow$ What will be the "typical" (average, expected) case?

Running Time and Complexity

"Run(ning)Time"

- the notation $T(n)$ suggest a "time", such as run(ning) time of an algorithm, which depends on the input (size) n
- in practice: we need a precise model how long each operation of our programmes takes $\rightarrow$ very difficult on real hardware!
- we will therefore determine the number of operations that determine the run time, such as:
  - number of comparisons (sorting, e.g.)
  - number of arithmetic operations (Fibonacci, e.g.)
  - number of memory accesses

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"Complexity"

- characterises how the run time depends on the input (size), typically expressed in terms of the Θ-notation
- "algorithm xyz has linear complexity" $\rightarrow$ run time is $\Theta(n)$

Average Case Complexity

Definition (expected running time)

Let $X(n)$ be the set of all possible input sequences of length n, and let $P\colon X(n) \to [0, 1]$ be a probability function such that $P(x)$ is the probability that the input sequence is x.
Then, we define

$$\bar{T}(n) = \sum_{x \in X(n)} P(x) T(x)$$

as the **expected running time** of the algorithm.

Comments:

- we require an exact probability distribution (for InsertionSort, we could assume that all possible sequences have the same probability)
- we need to be able to determine $T(x)$ for any sequence x (usually much too laborious to determine)

Average Case Complexity of Insertion Sort

Heuristic estimate:

- we assume that we need $\frac{j}{2}$ steps in every iteration:

$$\Rightarrow \bar{T}_{\mathrm{IS}}(n) \overset{(?)}{\approx} \sum_{j=2}^{n} \frac{j}{2} = \frac{1}{2} \sum_{j=2}^{n} j \in \Theta(n^2)$$

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- note: $\frac{j}{2}$ isn't even an integer ...

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- note: $\frac{j}{2}$ isn't even an integer …
- **Just considering the number of comparisons of the “average case” can lead to quite wrong results!**

in general $\quad E(T(n)) \neq T(\text{“}E(n)\text{”})$

Bubble Sort

```
BubbleSort(A:Array[1..n]) {
    for i from 1 to n do {
        for j from n downto i+1 do {
            if A[j] < A[j−1]
            then exchange A[j] and A[j−1]
        }
    }
}
```

Basic ideas:

- compare neighboring elements only
- exchange values if they are not in sorted order
- repeat until array is sorted (here: pessimistic loop choice)

Bubble Sort – Homework

Prove correctness of Bubble Sort:

- find invariant for i-loop
- find invariant for j-loop

Number of comparisons in Bubble Sort:

- best/worst/average case?

Part II

Mergesort and Quicksort

Mergesort

Basic Idea: divide and conquer

- **Divide** the problem into two (or more) subproblems:
  $\rightarrow$ split the array into two arrays of equal size
- **Conquer** the subproblems by solving them recursively:
  $\rightarrow$ sort both arrays using the sorting algorithm
- **Combine** the solutions of the subproblems:
  $\rightarrow$ merge the two sorted arrays to produce the entire sorted array

Combining Two Sorted Arrays: Merge

```
Merge (L:Array[1..p], R:Array[1..q], A:Array[1..n]) {
// merge the sorted arrays L and R into A (sorted)
// we presume that n=p+q
   i :=1; j :=1:
   for k from 1 to n do {
      if i > p
         then { A[k]:=R[j ]; j=j+1; }
      else if j > q
         then { A[k]:=L[ i ]; i :=i+1; }
      else if L[ i ] < R[j]
         then { A[k]:=L[ i ]; i :=i+1; }
         else { A[k]:=R[j ]; j :=j+1; }
   }
}
```

Correctness and Run Time of Merge

Loop invariant:

Before each cycle of the for loop:

- A has the k-1 smallest elements of L and R already merged, (i.e. in sorted order and at indices 1, . . . , k-1);
- L[i] and R[j] are the smallest elements of L and R that have not been copied to A yet
  (i.e. L[1..i-1] and R[1..j-1] have been merged to A)

Run time:

$$T_{\text{Merge}}(n) \in \Theta(n)$$

- for loop will be executed exactly n times
- each loop contains constant number of commands:
  - exactly 1 copy statement
  - exactly 1 increment statement
  - 1–3 comparisons

MergeSort

```
MergeSort(A:Array[1..n]) {
    if  n > 1 then {
        m := floor(n/2);
        create array L [1... m];
        for  i  from 1 to m do { L[i]  :=  A[i ];  }

        create array R [1... n−m];
        for  i  from 1 to n−m do { R[i] := A[m+i]; }

        MergeSort(L);
        MergeSort(R);

        Merge(L,R,A);
    }
}
```

Number of Comparisons in MergeSort

- Merge performs exactly n element copies on n elements
- Merge performs at most $c \cdot n$ comparisons on n elements
- MergeSort itself does not contain any comparisons between elements; all comparisons done in Merge

$\Rightarrow$ number of element-copy operations for the entire MergeSort algorithms can be specified by a recurrence (includes n copy operations for splitting the arrays):

$$C_{\mathrm{MS}}(n) = \begin{cases} 0 & \text{if} \quad n \leq 1 \\ C_{\mathrm{MS}}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + C_{\mathrm{MS}}\left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) + 2n & \text{if} \quad n \geq 2 \end{cases}$$

$\Rightarrow$ number of comparisons for the entire MergeSort algorithm:

$$T_{\mathrm{MS}}(n) \leq \begin{cases} 0 & \text{if} \quad n \leq 1 \\ T_{\mathrm{MS}}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T_{\mathrm{MS}}\left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) + cn & \text{if} \quad n \geq 2 \end{cases}$$

Number of Comparisons in MergeSort (2)

Assume $n = 2^k$, c constant:

$$
\begin{aligned}
T_{\mathrm{MS}}(2^k) &\le T_{\mathrm{MS}}\left(2^{k-1}\right) + T_{\mathrm{MS}}\left(2^{k-1}\right) + c \cdot 2^k \\
&\le 2\,T_{\mathrm{MS}}\left(2^{k-1}\right) + 2^k c \\
&\le 2^2\,T_{\mathrm{MS}}\left(2^{k-2}\right) + 2 \cdot 2^{k-1} c + 2^k c \\
&\le \ldots \\
&\le 2^k\,T_{\mathrm{MS}}\left(2^0\right) + 2^{k-1} \cdot 2^1 c + \ldots + 2^j \cdot 2^{k-j} c \\
&\qquad + \ldots + 2 \cdot 2^{k-1} c + 2^k c \\
&\le \sum_{j=1}^{k} 2^k c = ck \cdot 2^k = cn \log_2 n \in O(n \log n)
\end{aligned}
$$

Quicksort

Basic Idea: divide and conquer

- **Divide** the input array A[p..r] into parts A[p..q] and A[q+1 .. r], such that every element in A[q+1 .. r] is larger than all elements in A[p .. q].
- **Conquer:** sort the two arrays A[p..q] and A[q+1 .. r]
- **Combine:** if the divide and conquer steps are performed in place, then no further combination step is required.

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Partitioning using a **pivot element**:

- all elements that are smaller than the pivot element should go into the “smaller” partition (A[p..q])
- all elements that are larger than the pivot element should go into the “larger” partition (A[q+1..r])

Partitioning the Array (Hoare's Algorithm)

```
Partition (A:Array[p..r]) : Integer {
   // x is the pivot (chosen as first element):
   x := A[p];
   // partitions grow towards each other
   i := p−1; j := r+1; // ( partition boundaries)
   while true do { // i<j: partitions haven't met yet
      // leave large elements in right partition
      do { j:=j−1; } while A[j]>x;
      // leave small elements in left partition
      do { i:=i+1; } while A[i]<x;
      // swap the two first "wrong" elements
      if i < j
      then exchange A[i] and A[j];
      else return j;
   }
}
```

Time Complexity of Partition

How many statements are executed by the nested while loops?

Time Complexity of Partition

How many statements are executed by the nested while loops?

- monitor increments/decrements of i and j
- after $n := r - p$ increments/decrements, i and j have the same value

$\Rightarrow$ $\Theta(n)$ comparisons with the pivot

$\Rightarrow$ $O(n)$ element exchanges

Hence: $T_{\text{Part}}(n) \in \Theta(n)$

Implementation of QuickSort

```
QuickSort (A:Array[p..r ])
{
    if  p>=r then return;
    // only proceed, if A has at least 2 elements:
    q :=  Partition (A);
    QuickSort (A[p..q ]);
    QuickSort (A[q+1..r ]);
}
```

Homework:

- prove correctness of Partition
- prove correctness of QuickSort

Time Complexity of QuickSort

Best Case:

- assume that all partitions are split exactly into two halves:

$$T_{\text{QS}}^{\text{best}}(n) = 2T_{\text{QS}}^{\text{best}}\left(\frac{n}{2}\right) + \Theta(n)$$

- analogous to MergeSort:

$$T_{\text{QS}}^{\text{best}}(n) \in \Theta(n \log n)$$

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- analogous to MergeSort:

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Worst Case:

- Partition will always produce one partition with only 1 element:

$$\begin{aligned} T_{QS}^{\text{worst}}(n) &= T_{QS}^{\text{worst}}(n-1) + T_{QS}^{\text{worst}}(1) + \Theta(n) \\ &= T_{QS}^{\text{worst}}(n-1) + \Theta(n) = T_{QS}^{\text{worst}}(n-2) + \Theta(n-1) + \Theta(n) \\ &= \ldots = \Theta(1) + \ldots + \Theta(n-1) + \Theta(n) \in \Theta(n^2) \end{aligned}$$

Time Complexity of QuickSort – Special Cases?

What happens if:

- A is already sorted?

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What happens if:

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  $\rightarrow$ partition sizes always 1 and n-1 $\Rightarrow \Theta(n^2)$

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- one partition has always at most *a* elements (for a fixed *a*)?

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- partition sizes are always $n(1 - a)$ and na with $0 < a < 1$?

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- partition sizes are always $n(1-a)$ and na with $0 < a < 1$?
  $\rightarrow$ same complexity as best case $\Rightarrow \Theta(n \log n)$

Questions:

- What happens in the “usual” case?
- Can we force the best case?

Randomized QuickSort

```
RandPartition ( A: Array [p..r] ): Integer {
    // choose random integer i between p and r
    i := rand(p,r);
    // make A[i] the (new) Pivot element:
    exchange A[i] and A[p];
    // call Partition with new pivot element
    q := Partition (A);
    return q;
}

RandQuickSort ( A:Array [p..r] ) {
    if p >= r then return;
    q := RandPartition(A);
    RandQuickSort (A[p...q]);
    RandQuickSort (A[q+1 ..r]);
}
```

Time Complexity of RandQuickSort

Best/Worst-case complexity?

Time Complexity of RandQuickSort

Best/Worst-case complexity?

- RandQuickSort may still produce the worst (or best) partition in each step
- worst case: $\Theta(n^2)$
- best case: $\Theta(n \log n)$

Time Complexity of RandQuickSort

Best/Worst-case complexity?

- RandQuickSort may still produce the worst (or best) partition in each step
- worst case: $\Theta(n^2)$
- best case: $\Theta(n \log n)$

However:

- it is not determined which input sequence (sorted order, reverse order) will lead to worst case behavior (or best case behavior);
- any input sequence might lead to the worst case or the best case, depending on the random choice of pivot elements.

Thus: only the **average-case complexity** is of interest!

Average Case Complexity of RandQuickSort

Assumptions:

- we compute $\bar{T}_{\mathrm{RQS}}(A)$,
  i.e., the expected run time of RandQuickSort for a given input A
- `rand(p,r)` will return uniformly distributed random numbers
  (all pivot elements have the same probability)
- all elements of A have different size: $\mathrm{A[i]} \neq \mathrm{A[j]}$

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  i.e., the expected run time of RandQuickSort for a given input A
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  (all pivot elements have the same probability)
- all elements of A have different size: A[i] $\neq$ A[j]

Basic Idea:

- only count number of comparisons between elements of A
- let z_i be the i-th smallest element in A
- define

$$X_{ij} = \begin{cases} 1 & z_i \text{ is compared to } z_j \\ 0 & \text{otherwise} \end{cases}$$

- random variable $T_{\mathrm{RQS}}(A) = \sum_{i<j} X_{ij}$

Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$\bar{T}_{\text{RQS}}(A) = \mathrm{E}\left[\sum_{i<j} X_{ij}\right]$$

Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$\begin{aligned}\bar{T}_{\mathrm{RQS}}(A) &= \mathrm{E}\left[\sum\nolimits_{i<j} X_{ij}\right] \\ &= \sum\nolimits_{i<j} \mathrm{E}\left[X_{ij}\right]\end{aligned}$$

Average Case Complexity of RandQuickSort

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Average Case Complexity of RandQuickSort

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- suppose an element between z_i and z_j is chosen as pivot **before** z_i or z_j are chosen as pivots; then z_i and z_j are never compared

Average Case Complexity of RandQuickSort

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- suppose an element between z_i and z_j is chosen as pivot **before** z_i or z_j are chosen as pivots; then z_i and z_j are never compared
- if either z_i or z_j is chosen as the first pivot in the range $z_i, \ldots, z_j$, then z_i will be compared to z_j

Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$
\begin{aligned}
\bar{T}_{\mathrm{RQS}}(A) &= \mathrm{E}\left[\sum\nolimits_{i<j} X_{ij}\right] \\
&= \sum\nolimits_{i<j} \mathrm{E}\left[X_{ij}\right] \\
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$$

- suppose an element between z_i and z_j is chosen as pivot **before** z_i or z_j are chosen as pivots; then z_i and z_j are never compared
- if either z_i or z_j is chosen as the first pivot in the range $z_i, \ldots, z_j$, then z_i will be compared to z_j
- this happens with probability

$$\frac{2}{j-i+1}$$

Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$\bar{T}_{\mathrm{RQS}}(A) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j-i+1}$$

Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$
\begin{aligned}
\bar{T}_{\mathrm{RQS}}(A) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{1}{k}
\end{aligned}
$$

Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$\begin{aligned}\bar{T}_{\mathrm{RQS}}(A) &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{j-i+1}\\ &= \sum_{i=1}^{n-1}\sum_{k=2}^{n-i+1}\frac{1}{k}\\ &\le 2\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{1}{k}\end{aligned}$$

Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$
\begin{aligned}
\bar{T}_{\mathrm{RQS}}(A) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{1}{k} \\
&\le 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \\
&= 2nH_n
\end{aligned}
$$

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&\leq 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \\
&= 2nH_n \\
&= O(n \log n)
\end{aligned}
$$

Part III

Outlook: Optimality of Comparison Sorts

Are Mergesort and Quicksort optimal?

Definition

Comparison sorts are sorting algorithms that use only comparisons (i.e. tests as $\leq, =, >, \dots$) to determine the relative order of the elements.

Examples:

- InsertSort, BubbleSort
- MergeSort, (Randomised) Quicksort

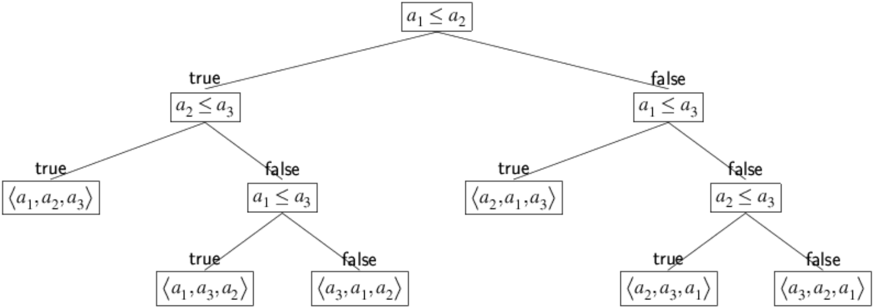
Question:

Is $T(n) \in \Theta(n \log n)$ the best we can get (in the worst/average case)?

Decision Trees

Definition

A **decision tree** is a binary tree in which each internal node is annotated by a comparison of two elements.
The leaves of the decision tree are annotated by the respective permutations that will put an input sequence into sorted order.



Decision Trees – Properties

Each comparison sort can be represented by a decision tree:

- a path through the tree represents a sequence of comparisons
- sequence of comparisons depends on results of comparisons
- can be pretty complicated for Mergesort, Quicksort, ...

A decision tree can be used as a comparison sort:

- if every possible permutation is annotated to at least one leaf of the tree!
- if (as a result) the decision tree has at least n! (distinct) leaves.

A Lower Complexity Bound for Comparison Sorts

- A binary tree of height h (h the length of the longest path) has at most 2^h leaves.
- To sort n elements, the decision tree needs $n!$ leaves.

Theorem

Any decision tree that sorts n elements has height $\Omega(n \log n)$.

Proof:

- h comparisons in the worst case are equivalent to a decision tree of height h
- with h comparisons, we can sort n elements (at best), if

$$n! \leq 2^h \quad \Leftrightarrow \quad h \geq \log(n!) \in \Omega(n \log n)$$

- because:

$$h \geq log(n!) \geq \log\left(n^{n/2}\right) = \frac{n}{2} \log n$$

Optimality of Mergesort and Quicksort

Corollaries:

- MergeSort is an optimal comparison sort in the worst/average case
- QuickSort is an optimal comparison sort in the average case

Consequences and Alternatives:

- comparison sorts can be faster than MergeSort, but only by a constant factor
- comparison sorts can not be asymptotically faster
- sorting algorithms might be faster, if they can exploit additional information on the size of elements
- examples: **BucketSort**, CountingSort, RadixSort

Part IV

Bucket Sort – Sorting Beyond "Comparison Only"

Bucket Sort

Basic Ideas and Assumptions:

- pre-sort numbers in buckets that contain all numbers within a certain interval
- hope (assume) that input elements are evenly distributed and thus uniformly distributed to buckets
- sort buckets and concatenate them

Requires “Buckets”:

- can hold arbitrary numbers of elements
- can insert elements efficiently: in $O(1)$ time
- can concatenate buckets efficiently: in $O(1)$ time
- remark: linked lists will do

Implementation of BucketSort

```
BucketSort (A:Array[1..n]) {

    Create Array B[0..n−1] of Buckets;
    // assume all Buckets B[i] are empty at first

    for i from 1 to n do {
        insert A[i] into Bucket B[floor(n * A[i])];
    }

    for i from 0 to n−1 do {
        sort Bucket B[i] ;
    }

    concatenate Buckets B[0], B[1], ..., B[n−1] into A
}
```

Number of Operations of BucketSort

Operations:

- n operations to distribute n elements to buckets
- plus effort to sort all buckets

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- plus effort to sort all buckets

Best Case:

- if each bucket gets 1 element, then $\Theta(n)$ operations are required

Worst Case:

- if one bucket gets all elements, then $T(n)$ is determined by the sorting algorithm for the buckets

Bucketsort – Average Case Analysis

- probability that bucket i contains k elements:

$$P(n_i = k) = \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k}$$

- expected mean and variance for such a distribution:

$$E[n_i] = n \cdot \frac{1}{n} = 1 \qquad \mathrm{Var}[n_i] = n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)$$

- InsertionSort for buckets $\Rightarrow \leq cn^2 \in O(n_i^2)$ operations per bucket
- expected operations to sort one bucket:

$$\bar{T}(n_i) \leq \sum_{k=0}^{n-1} P(n_i = k) \cdot ck^2 = cE[n_i^2]$$

Bucketsort – Average Case Analysis (2)

- theorem from statistics:

$$E[X^2] = E[X]^2 + \mathrm{Var}(X)$$

- expected operations to sort one bucket:

$$\bar{T}(n_i) \le cE[n_i^2] = c\left(E[n_i]^2 + \mathrm{Var}[n_i]\right) = c\left(1^2 + 1 - \frac{1}{n}\right) \in \Theta(1)$$

- expected operations to sort all buckets:

$$\bar{T}(n) = \sum_{i=0}^{n-1} \bar{T}(n_i) \le c\sum_{i=0}^{n-1}\left(2 - \frac{1}{n}\right) \in \Theta(n)$$

(note: expected value of the sum is the sum of expected values)