Fundamental Algorithms

Chapter 9: Weighted Graphs

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Weighted Graphs

Definition (Weighted Graph)

A **weighted graph** $G = (V, E)$ is attributed by a function w that assigns a weight $w(e)$ to each edge $e \in E$.

Comments

- typically: $w(e) > 0$ or $w(e) \geq 0$ (but negative weights possible)
- we will consider weighted graphs with $w : E \to \mathbb{N}$
- notation: we will also write $w(V, W)$, instead of $w((V, W))$, for the weight $w(e)$ of the edge $e = (V, W)$
- examples: traffic networks, costs for routing, etc.

Shortest Path

Definition (Length of a Path)

The length of a path $p = (V_0, V_1), (V_1, V_2), \ldots, (V_{n-1}, V_n)$ in a weighted graph is defined as

$$\overline{w}(p) := \sum_{j=1}^{n} w(V_{j-1}, V_j).$$

Definition (Distance between Vertices)

The **distance** $d(V, W)$ between two vertices V and W is defined as the length of the shortest path $p = (V_0, V_1), (V_1, V_2), \ldots, (V_{n-1}, V_n)$ that connects V and W:.

$$d(V, W) = \min\Big\{\, \overline{w}(p) \;:\; p = (V_0, V_1), (V_1, V_2), \ldots, (V_{n-1}, V_n), \\ \forall j\colon (V_{j-1}, V_j) \in E, V = V_0, W = V_n \,\Big\}$$

All-Pairs Shortest Path

For non-weighted graphs: (try this at home!)

BF-traversal finds the shortest path from a starting node to all connected nodes.

- → is there an efficient algorithm to find the shortest path from all nodes to all other nodes? ("all-pairs shortest path")
- → is there an efficient algorithm to find which nodes are connected by a path of length l?
- → is there an efficient algorithm to find which nodes are connected by only the first k nodes? (assuming an ordering of the nodes)

For weighted graphs:

Generalize the last idea for weighted graphs

- → Incrementally construct shortest paths from nodes connected by only the first k nodes
- → We will implement the algorithm for **directed graphs** (modifying it for undirected graphs is straightforward)

Floyd's Algorithm

```
Floyd_basic(W:Array[1..n,1..n]) {
  ! Input: weight/adjacency matrix W
  ! assume: W[i,j] = inf, if i not connected to j
  ! Output: W[i,j] shortest part from i to j

  for k from 1 to n do
    ! check for all (i,j) whether a shorter path exists
    ! that runs through vertex k
    for i from 1 to n do
      for j from 1 to n do
        W[i,j] = min( W[i,k]+W[k,j], W[i,j] )
      end do
    end do
  end do
}
```

Floyd's Algorithm (2)

Disadvantages of Floyd_basic:

- input array W is overwritten
- we get the length of the shortest path, but not the path itself!

```
Floyd(W:Array[1..n,1..n],
      S:Array[1..n,1..n], P:Array[1..n,1..n]) {
  ! Output: S will contain lengths
  ! P allows to reconstruct shortest path
  for i from 1 to n do
    for j from 1 to n do
      S[i,j] = W[i,j]
      P[i,j] = 0
    end do
  end do
```

Floyd's Algorithm (3)

```
! main loop of Floyd():
for k from 1 to n do
  for i from 1 to n do
    for j from 1 to n do
      if S[i,k] + S[k,j] < S[i,j] then
        S[i,j] = S[i,k] + S[k,j];
        ! memorize connection via k
        P[i,j] = k;
      end if
    end do
  end do
}
```

Use array P to reconstruct shortest path:

- P[i,j] indicates that shortest path runs through vertex k
- check P[i,k] and P[k,j] for further info

Floyd's Algorithm – Correctness

Ingredients:

- **Optimality Principle:**
  If the shortest path between nodes A and B visits a node C, then this path consists of the shortest paths between A and C, and between C and B.
- **No cycles:**
  The shortest path between any two nodes does not contain a cycle, i.e., contains any node at most once.
  $\rightarrow$ while edges are allowed to have negative weights, cycles must not lead to negative weight
- **Loop Invariant** for the k-loop:
  At entry of the k-loop, $S[i,j]$ contains (for every pair i,j) the length of the shortest path between i and j that only visits nodes with index smaller than k.

Floyd's Algorithm on the PRAM

```
FloydPRAM(W: Array[1..n,1..n]) {
  for k from 1 to n do
    for i from 1 to n do in parallel
      for j from 1 to n do in parallel
        if W[i,k]+W[k,j] < W[i,j]
        then W[i,j] = W[i,k]+W[k,j]
      end do
    end do
  end do
}
```

Classify concurrent/exclusive read/write?

- **concurrent read** to row $W[*,k]$ and column $W[k,*]$

Dijkstra's Algorithm for Shortest Paths

Problem setting: "single-source shortest path"

- given is a directed graph $G = (V, E)$ and a start vertex $r \in V$
- we want to compute the shortest path from r to each vertex in G that is reachable from r
  $\rightarrow$ this is a **spanning tree** of shortest paths

Idea: "Greedy Algorithm"

- maintain a spanning tree S of vertices and "explored" shortest paths
- maintain a set $Q = V \setminus S$ of unexplored vertices
- for each $v \in Q$, determine the shortest path to v that can be obtained by adding a single edge to the spanning tree S
- add v_{min} (with shortest path) to S and update Q
- repeat until all vertices are in the explored path

Dijkstra's Algorithm – Implementation

Spanning Tree S of Shortest Paths

- use an array Parent[1..n] for the n vertices
- Parent[i] contains the parent of vertex i in the spanning tree

Set Q of Unexplored Vertices

- accompanied by an array Dist [1.. n]
- Dist[i] contains the shortest path to vertex i that adds only one edge to S
- we will need to update Dist [1.. n] after each change of Q
- for vertices $i \notin Q$, Dist[i] is the length of the shortest path (i.e., they will not be further considered; therefore weights must not be negative!)

Dijkstra's Algorithm – Implementation (2)

```
Dijkstra(W:Array[1..n,1..n], r:Node) {
  ! initialise data structures
  Array Parent[1..n];
  Array Dist[1..n];
  for i from 1 to n do
    Dist[i] = inf;
  end do;
  ! init Parent and Dist for root r:
  Parent[r] = 0;
  Dist[r] = 0;
  ! init sets of explored and unexplored vertices
  Set S = {};
  Set Q = {1, .., n};
  ! ... to be continued ...
```

Dijkstra's Algorithm – Implementation (3)

```
    ! main loop of Dijkstra(...)
    while Q <> {} do
      ! remove node with smallest Dist[] from Q
      X = removeSmallest(Q,Dist);
      S = union(S,X);
      ! X is added to S, thus update Dist:
      forall (X,V) in X.edges do
        if V in S then continue;
        ! update neighbours of X that are not in S:
        d := Dist[X.key] + W[X.key,V.key);
        if d < Dist[V.key] then
           Dist[V.key] := d;
           Parent[V.key] := X.key ;
        end if
      end do;
    end while;
}
```

Dijkstra's Algorithm – Comments

- Why do we not update Dist[X.key] and Parent[X.key]?

$\rightarrow$ this was already set in the previous iteration of the while-loop

- how do we obtain the shortest path?

$\rightarrow$ via the Parent[] array:

```
shortestPath(key:Int) : List {
  if Parent[key] = 0
  then return [key]
  else return append(shortestPath(Parent[key]), key);
  end if;
}
```

Dijkstra's Algorithm – Complexity

Priority Queues:

- How is the function removeSmallest implemented?
- Idea: sort elements of Q according to array Dist
- ToDo: Update sorting of Q after changes to Dist

```
if d < Dist[V.key] then
    Parent[V.key] := X.key ;
    Dist[V.key] := d;
    updateSorting(Q, Dist ,V);
end if
```

- integrated data structure for such purposes: **priority queue**

Complexity of Dijkstra's Algorithm:

- a complexity of $\Theta(|E| + |V| \log |V|)$ is possible
- for dense graphs, $|E| \in \Theta|V|^2$, the complexity is thus $\Theta(|V|^2)$

Dijkstra – Single Source, Single Destination

Single Source, All Destinations:

- we can terminate Dijkstra's Algorithm after the destination node has been removed from Q:

```
X = removeSmallest(Q, Dist);
if X = destination then return X;
```

- otherwise Dijkstra's Algorithm finds the shortest path from the source to all nodes in the graph.

Question:

Can Dijkstra's Algorithm be improved, if the shortest path to only one specific destination is wanted?

- or more general: is there a better algorithm to solve the single-source-single-destination problem?

$\rightarrow$ there is no algorithm known that is asymptotically faster

Minimum Spanning Tree

Definition (Minimum Spanning Tree)

A spanning tree $T = (V, E)$ is called a **minimum spanning tree** for the graph $G = (V, E')$, if the sum of the weights of all edges of T is minimal (among all possible spanning trees).

Towards an Algorithm:

- Dijkstra's Algorithm computes a spanning tree of shortest paths
- Idea: modify Dijkstra's "greedy approach"
  $\rightarrow$ successively add edges to a subtree
- minimise total weight of edges instead of path lengths
  $\rightarrow$ add node that is closest to the current subtree

$\Rightarrow$ **Prim's Algorithm**

Prim's Algorithm

```
Prim(W: Array[1..n,1..n], r:Node) {
  ! initialise data structures
  Array Parent[1..n];
  Array Nearest[1..n]; ! replaces Dist
  for i from 1 to n do
    Nearest[i] = inf;
  end do;
  ! init Parent and Dist for root r:
  Parent[r] = 0;
  Nearest[r] = 0;
  ! init sets of explored and unexplored vertices
  Set S = {};
  Set Q = {1, .., n};
  ! ... to be continued ...
```

Prim's Algorithm (2)

```
    ! main loop of Prim(...)
    while Q <> {} do
      ! remove nearest node from Q
      X = removeNearest(Q,Nearest);
      S = union(S,X);
      ! X is added to S, thus update Nearest:
      forall (X,V) in X.edges do
        if V in S then continue;
        ! update neighbours of X that are not in S:
        if W[X.key,V.key] < Nearest[V.key] then
          Nearest[V.key] := W[X.key,V.key];
          Parent[V.key] := X.key ;
        end if
      end do;
    end while;
}
```